Physics 531
Problem Set #2 – Due Wed. August 31, 2011

Problem 1: Hydrogenic atoms and characteristic scales.

Consider the "hydrogenic" atoms - that is bound-states of two oppositely charged particles:

(i) The hydrogen atom: Binding of an electron and proton.

(ii) Heavy ion: Single electron bound to a nucleus of mass *M*, charge *Ze* (say Z=50).

(iii) Muonium: Muon bound to a proton

(iv) Positronium: Bound state of an electron and a positron (anti-electron)

(a) For each, using the charges, *reduced* mass, and the unit $\hbar$, determine the characteristic scales of:

Length, energy, time, momentum, internal electric field, and electric dipole moment. Please give numerical values as well as the expressions in terms of the fundamental constants.

(b) Now add the speed of light *c* into the mix. Find characteristic velocity in units of *c*, magnetic field, and magnetic moment. Show that for the particular case of hydrogen the characteristic velocity is $v / c = \alpha = \frac{e^2}{\hbar c} (cgs) \approx \frac{1}{137}$, the "fine-structure" constant, and that the Bohr radius, Compton wavelength, and "classical electron radius", differ by powers of α according to,

$$r_{class} = \alpha \lambda_{compton} = \alpha^2 a_0$$

(c) What is the characteristic magnetic field and magnetic dipole moment?

Problem 2: Radial Expectation Values for Hydrogen

(a) By brute force, using generating functions for Laguerre polynomials, show that mean radius a one-electron atom in the hydrogenic orbital $|n,l,m\rangle$ is

$$\langle r\rangle_{nl} = n^2\frac{a_0}{Z}\left[1+\frac{1}{2}\left(1-\frac{l(l+1)}{n^2}\right)\right] \text{ (independent of q-number } m\text{)}$$

(b) For "circular" states (the ones with zero radial momentum, n_r=0), and in the "correspondence limt" ($n\to\infty$) show that we retrieve Bohr's result,

$$\langle r\rangle \to n^2\frac{a_0}{Z}.$$

Though any expectation value can be calculated by tedious method in part (a), a trick to due Feyman and Hellman, saves a lot of work (note this was part of Feyman's undergrad thesis!).
The radial Hamiltonian is a function of various "parameters", $m_e, e, l \equiv \xi$,

$$\hat{H}(m_e,e,l) = \frac{-\hbar^2}{2m_e}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2m_e r^2} - \frac{Ze^2}{r}$$

Mathematically, is well defined for arbitrary assignment or real numbers to any ξ.

(c) Defining the radial eigenstate as $\hat{H}(\xi)|n_r,\xi\rangle = E_{n_r}(\xi)|n_r,\xi\rangle = -\frac{1}{2(n_r+l+1)^2}\frac{Z^2me^4}{\hbar^2}|n_r,\xi\rangle$,

show that $\langle n_r,\xi|\frac{\partial\hat{H}(\xi)}{\partial\xi}|n_r,\xi\rangle = \frac{\partial E_{n_r}(\xi)}{\partial\xi}$ (Feyman-Hellman theorem)

(d) Using the Feynman-Hellman theorem, show that

- $\left\langle\frac{1}{r}\right\rangle_{n,l} = \frac{Z}{a_0 n^2}$ (use $\xi = e^2$). Relate this to the Viral Theorem.
- $\left\langle\frac{1}{r^2}\right\rangle_{n,l} = \frac{Z^2}{a_0^2}\frac{1}{n^3(l+1/2)}$ (use $\xi = l$).
- $\left\langle\frac{1}{r^3}\right\rangle_{n,l} = \frac{Z}{a_0}\frac{1}{l(l+1)}\left\langle\frac{1}{r^2}\right\rangle_{n,l}$.

For this final case prove and then use the expectation value of the commutator,

$$\left\langle\left[\frac{d}{dr},\hat{H}(\xi)\right]\right\rangle_{nl} = 0.$$

Problem 3: Finite nuclear size effect

The Coulomb potential in a hydrogen-like atom is modified at short distances because of the finite size of the nucleus. As a simple model, let us assume that the total charge *Ze* is distributed as a uniform sphere of radius r_p.

(a) Using elementary electrostatics, what is the potential *V(r)* seen by a particle bound in the field of the extended proton.

(b) Treating the difference between the point Coulomb potential and the one for a finite nucleus as a perturbation, calculate the first order energy change in the ground state of a hydrogen-like atom consisting of a nucleus of total charge *Ze* and a particle of mass *m* and charge -*e*, without any assumptions about the relative size of the atom and nucleus.

(c) For *Z*=82 and $r_p = 8 \times 10^{-13}$ cm (lead nucleus), find the ratio of the energy shift to the unperturbed binding energy for a hydrogenic atom consisting of the lead nucleus plus one electron and then for one muon ($m_\mu = 206.77\, m_e$). Also calculate the ratio of the mean atomic radius of the unperturbed atom to the nuclear radius in both instances.

(d) Extra credit: The result of the muonic atom is not reliable - why? Develop a better answer for the ground state of the muon in the field of the lead nucleus.

Comment: The size of proton radius as measured in atomic physics is one of today's hottest topics. In muonium (muon + proton), relativistic effects also play an important role. By measuring the Lamb shift, one can deduce the proton radius. In a recent experiment, and based on QED calculations, it was found that r_p =5 0.84184(67) fm, which differs by 5.0 standard deviations from the CODATA compilation of physical constants value of 0.8768(69) fm.

See, "The size of the proton" by Randolf Pohl *et. al., Nature* **466**, 213 (2010).