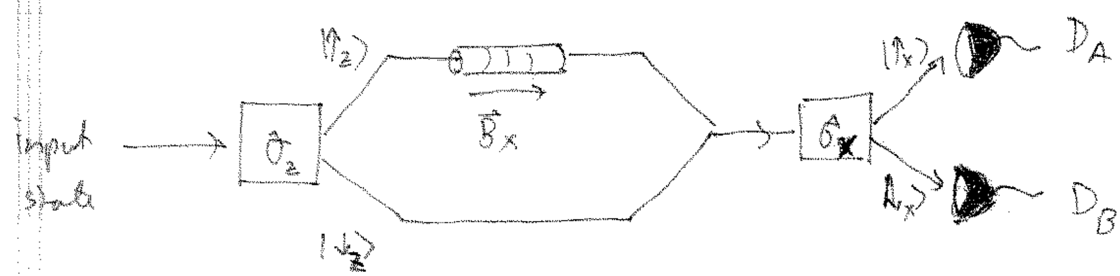
Physics 566

P.S. #1 Solutions

Problem 1: Spin interferometer and coherence



(a) Consider an arbitrary pure state

$$|\psi\rangle = c_\uparrow |\uparrow_z\rangle + c_\downarrow |\downarrow_z\rangle$$

The spin $|\uparrow_z\rangle$ branch gets rotated by the operator $\hat{R}_x(\phi) = e^{-i\phi\frac{\hat{\sigma}_x}{2}} = \cos(\frac{\phi}{2})\hat{\mathbb{1}} - i\sin(\frac{\phi}{2})\hat{\sigma}_x$

$\Rightarrow$ Right before the final beam splitter

$$|\psi'\rangle = c_\uparrow \hat{R}_x |\uparrow_z\rangle + c_\downarrow |\downarrow_z\rangle$$

$\Rightarrow$ Probability of detector D_B firing

$$P_{D_B} = |\langle\uparrow_x|\psi'\rangle|^2$$

$$= |c_\uparrow \langle\uparrow_x|\hat{R}_x|\uparrow_z\rangle + c_\downarrow \langle\uparrow_x|\downarrow_z\rangle|^2$$

(Next page)

Aside: • $\langle \uparrow_x | \hat{R}_x(\phi) | \uparrow_z \rangle =$

$$= \cos\left(\frac{\phi}{2}\right) \langle \uparrow_x | \uparrow_z \rangle - i \sin\left(\frac{\phi}{2}\right) \langle \uparrow_x | \hat{\sigma}_x | \uparrow_z \rangle$$

$$= \left(\cos\frac{\phi}{2} - i \sin\frac{\phi}{2}\right) \langle \uparrow_x | \uparrow_z \rangle \qquad \left(\text{using } \langle \uparrow_x | \hat{\sigma}_x = \langle \uparrow_x | \right)$$

$$= e^{-i\phi/2} \langle \uparrow_x | \uparrow_z \rangle$$

• Recall $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$

$$\Rightarrow \langle \uparrow_x | \uparrow_z \rangle = \langle \; | \downarrow_z \rangle = \frac{1}{\sqrt{2}}$$

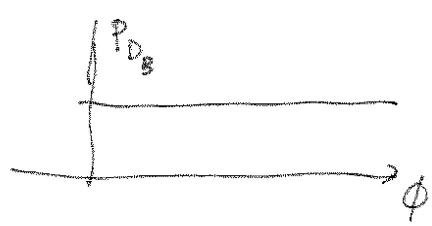
$$\therefore \quad P_{D_B} = \frac{1}{2} \left| c_\uparrow e^{-i\phi/2} + c_\downarrow \right|^2$$

$$\boxed{P_{D_B} = \frac{1}{2}\left(|c_\uparrow|^2 + |c_\downarrow|^2 + 2\,\mathrm{Re}\left(c_\uparrow c_\downarrow^* e^{-i\phi/2}\right)\right)}$$

Examples:

(i) $|\psi\rangle = |\uparrow_z\rangle \qquad c_\uparrow = 1, \; c_\downarrow = 0$

$$\Rightarrow \boxed{P_{D_B} = \frac{1}{2}}$$



(ii) $|\psi\rangle = |\downarrow_z\rangle \qquad c_\uparrow = 0, \; c_\downarrow = 1$

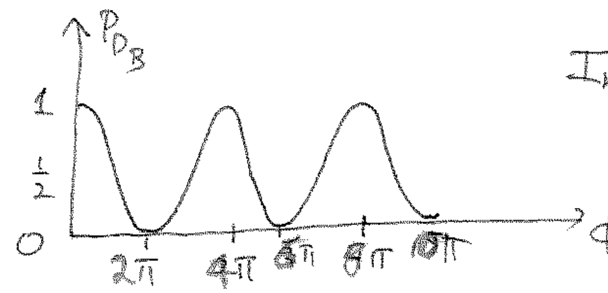
$$\Rightarrow \boxed{P_{D_B} = \frac{1}{2}}$$

These case show no interference because we know which path

(iii) $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle + \frac{1}{\sqrt{2}}|\downarrow_z\rangle, \quad c_\uparrow = c_\downarrow = \frac{1}{\sqrt{2}}$

$$\Rightarrow P_{D_B} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} + \mathrm{Re}\left(e^{-i\phi/2}\right)\right)$$

$$= \frac{1+\cos\frac{\phi}{2}}{2} = \cos^2\frac{\phi}{4}$$

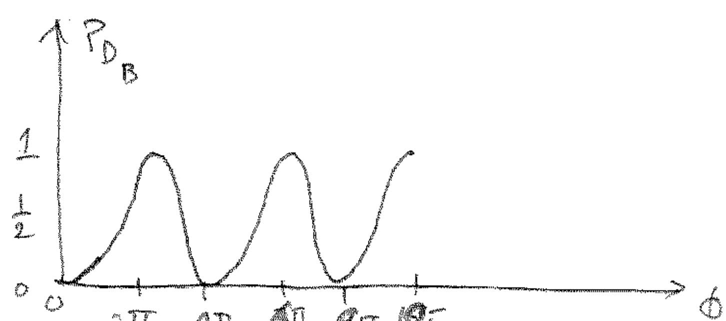


Interference "fringes"
(Note half period due to SU(2) rotation)

(iv) $|\downarrow_x\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle - \frac{1}{\sqrt{2}}|\downarrow_z\rangle, \quad c_\uparrow = -c_\downarrow = \frac{1}{\sqrt{2}}$

$$\Rightarrow P_{D_B} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} - \mathrm{Re}\left(e^{-i\phi/2}\right)\right)$$

$$= \frac{1-\cos\frac{\phi}{2}}{2} = \sin^2\frac{\phi}{4}$$



(b) We consider now mixed states

Generally $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

This is a ~~statistical~~ statistical mixture of states $\{|\psi_i\rangle\}$, not a coherent superposition of states. We should think of it "classically" → We have one of the set $\{|\psi_i\rangle\}$ we just don't know which one

$$\Rightarrow P_{D_B} = \sum_i \underbrace{p_i}_{\text{Prob of } |\psi_i\rangle} \underbrace{|\langle\uparrow_x|\psi_i\rangle|^2}_{\text{probability of } |\uparrow_x\rangle \text{ given } |\psi_i\rangle}$$

(i) $\hat{\rho} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|$

Given $|\uparrow_z\rangle \Rightarrow P_{D_B}^{(\uparrow_z)} = \frac{1}{2}$

$|\downarrow_z\rangle \Rightarrow P_{D_B}^{(\downarrow_z)} = \frac{1}{2}$

$$\Rightarrow P_{D_B}^{total} = P_{\uparrow_z} P_{D_B}(\uparrow_z) + P_{\downarrow_z} P_{D_B}(\downarrow_z)$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{2}$$

(ii) $\hat{\rho} = \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| + \frac{1}{2}|\downarrow_x\rangle\langle\downarrow_x|$

Given $|\uparrow_x\rangle \Rightarrow P_{D_B}^{(\uparrow_x)} = \cos^2\frac{\phi}{2}$

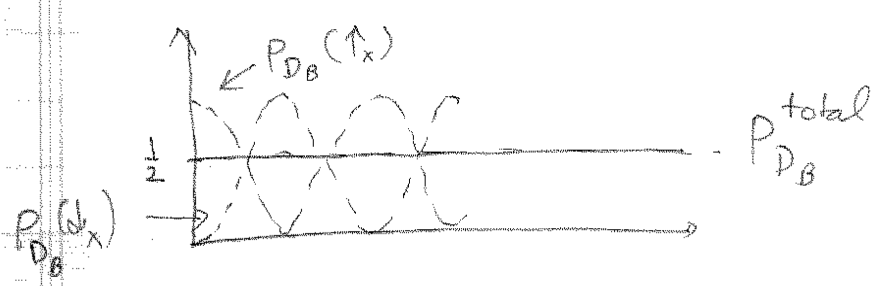
$|\downarrow_x\rangle \Rightarrow P_{D_B}^{(\downarrow_x)} = \sin^2\phi/2$

$$\Rightarrow P_{D_B}^{total} = P_{\uparrow_x} P_{D_B}(\uparrow_x) + P_{\downarrow_x} P_{D_B}(\downarrow_x)$$

$$= \frac{1}{2}\cos^2\phi/2 + \frac{1}{2}\sin^2\phi/2$$

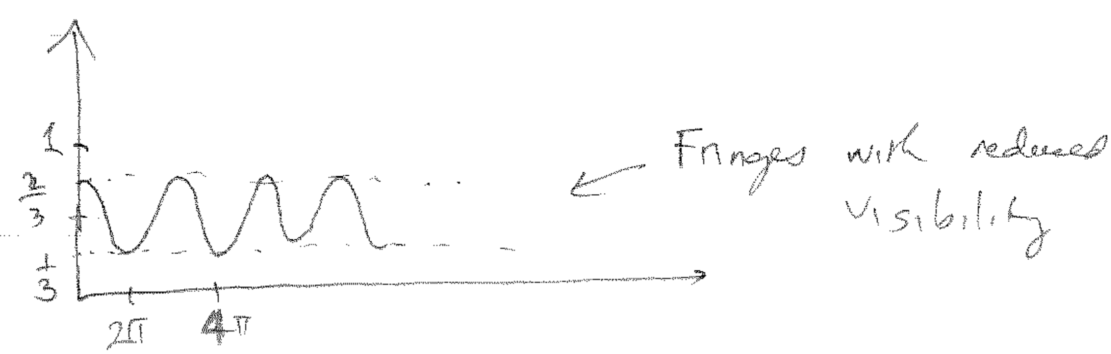
$$= \frac{1}{2}(\cos^2\phi/2 + \sin^2\phi/2) = \frac{1}{2}$$

$\Rightarrow$ For a completely mixed state the interference is removed



(iii) $\hat{\rho} = \frac{1}{3}|\uparrow_x\rangle\langle\uparrow_x| + \frac{2}{3}|\downarrow_x\rangle\langle\downarrow_x|$

$$P_{D_B} = \frac{1}{3}\cos^2\phi + \frac{2}{3}\sin^2\phi = \frac{1}{3} + \frac{1}{3}\sin^2\phi$$



A partial mixed state of two states which show interference shows fringes with reduced visibility.

Problem 2

(a) $$\mathrm{Tr}(|\phi\rangle\langle\xi|) = \sum_n \langle n|\phi\rangle\langle\xi|n\rangle \quad \text{for basis } \{n\}$$

$$= \sum_n \langle\xi|n\rangle\langle n|\phi\rangle = \langle\xi|\underbrace{\left(\sum_n |n\rangle\langle n|\right)}_{=\mathbb{1}}|\phi\rangle$$

$$= \langle\xi|\phi\rangle$$

~~Trace turns outer product to inner product~~

(a) Consider a statistical mixture

$$\hat{\rho} = P_+ |+_z\rangle\langle +_z| + P_- |-_z\rangle\langle -_z|$$

where $P_\pm = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right)$

Density matrix $\rho_{ij} = \langle i|\hat{\rho}|j\rangle$

In basis $|\pm_z\rangle$ $$\hat{\rho} \doteq_z \frac{1}{2}\begin{pmatrix} 1+\frac{1}{\sqrt{2}} & 0 \\ 0 & 1-\frac{1}{\sqrt{2}} \end{pmatrix}$$

In basis $\left(|\pm_x\rangle = \frac{1}{\sqrt{2}}(|+_z\rangle \pm |-_z\rangle)\right)$

$$\hat{\rho} \doteq_x \begin{pmatrix} \langle +_x|\hat{\rho}|+_x\rangle & \langle +_x|\hat{\rho}|-_x\rangle \\ \langle -_x|\hat{\rho}|+_x\rangle & \langle -_x|\hat{\rho}|-_x\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

Note: In this basis the density operator has off-diagonal elements. Nonetheless, it is a mixed state:

$$\mathrm{Tr}(\hat{\rho}^2) = \frac{3}{4}$$

The Bloch vector can be seen immediately from the form in the z-basis.

$$\mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_x) = \mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_y) = 0$$

$$\mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_z) = P_+ - P_- = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{\vec{Q} = \frac{1}{\sqrt{2}}\,\vec{e}_z} \qquad \text{mixed state } |\vec{Q}| < 1$$

(b) Now we have a state

$$\hat{\rho} = \frac{1}{2}|+_{n_1}\rangle\langle +_{n_1}| + \frac{1}{2}|+_{n_2}\rangle\langle +_{n_2}|$$

where $|+_n\rangle\langle +_n| = \frac{1}{2}(\hat{\mathbb{1}} + \vec{e}_n \cdot \hat{\vec{\sigma}})$ from Prob 1

$$\vec{e}_{n_{1,2}} = \frac{1}{\sqrt{2}}(\vec{e}_z \pm \vec{e}_x)$$

$$\Rightarrow \hat{\rho} = \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{4}(\vec{e}_{n_1} + \vec{e}_{n_2}) \cdot \hat{\vec{\sigma}}$$

$$= \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{4}\left(\frac{2}{\sqrt{2}}\vec{e}_z\right) \cdot \hat{\vec{\sigma}}$$

$$= \frac{1}{2}\left(\hat{\mathbb{1}} + \frac{1}{\sqrt{2}}\vec{e}_z\right) \cdot \vec{\sigma} \;\underset{z}{=}\; \begin{bmatrix} \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & 0 \\ 0 & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) \end{bmatrix}$$

Same as $\hat{\rho}$ in part (b)!

Moral of the story: The ensemble decomposition is not unique. In fact, we can take any density matrix for a two-level system, described uniquely in terms of its Bloch vector $\vec{Q}$ and decompose it in terms of an ensemble of any two pure states described by unit vector $\vec{e}_n$ with probability p_n if $\vec{Q} = p_{n_1}\vec{e}_{n_1} + p_{n_2}\vec{e}_{n_2}$.

(c) Two statistical mixtures

$$\hat{\rho}_1 = \sum_n p_n |+_n\rangle\langle +_n|$$

$$\hat{\rho}_2 = \sum_m q_m |+_m\rangle\langle +_m|$$

Aside: $|+_n\rangle\langle +_n| = \frac{1}{2}(\hat{\mathbb{1}} + \hat{\sigma}_n)$ $\left(\begin{array}{l}\text{where} \\ \hat{\sigma}_n = \vec{e}_n \cdot \hat{\vec{\sigma}}\end{array}\right)$

Projector

$$\Rightarrow \hat{\rho}_1 = \underbrace{\left(\sum_n p_n\right)}_{=1} \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{2}\underbrace{\left(\sum_n p_n \vec{e}_n\right)}_{\vec{Q}_1} \cdot \hat{\vec{\sigma}}$$

$$\Rightarrow \hat{\rho}_1 = \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{2}\vec{Q}_1 \cdot \hat{\vec{\sigma}}$$

Similarly $\hat{\rho}_2 = \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{2}\vec{Q}_2 \cdot \hat{\vec{\sigma}}$

where $\vec{Q}_2 = \sum_m q_m \vec{e}_m$

Thus $\hat{\rho}_1 = \hat{\rho}_2 \iff \vec{Q}_1 = \vec{Q}_2$

<u>Problem 3:</u> Ambiguity of ensemble decomposition

Let $\hat{\rho}_1 = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\hat{\rho}_2 = \sum_j q_j |\phi_j\rangle\langle\phi_j|$

<u>Prove</u> $\hat{\rho}_1 = \hat{\rho}_2$ iff $\sqrt{q_j}\,|\phi_j\rangle = \sum_i U_{ji} \sqrt{p_i}\, |\psi_i\rangle$

where U_{ji} are elements of unitary matrix.

<u>Proof:</u>

For convenience, define $|\bar{\phi}_j\rangle \equiv \sqrt{q_j}\,|\phi_j\rangle$

$|\bar{\psi}_i\rangle \equiv \sqrt{p_i}\,|\psi_i\rangle$

so $\langle\bar{\phi}_j|\bar{\phi}_j\rangle = q_j$ $\langle\bar{\psi}_i|\bar{\psi}_i\rangle = p_i$

(i) Assume $|\bar{\phi}_j\rangle = \sum_i U_{ji} |\bar{\psi}_i\rangle$ U_{ji} elements of unitary matrix

Consider $\hat{\rho}_2 = \sum_j |\bar{\phi}_j\rangle\langle\bar{\phi}_j| = \sum_{ijk} U^*_{jk} U_{ji} |\bar{\psi}_i\rangle\langle\bar{\psi}_k|$

<u>Aside:</u> $(U_{jk})^* = U^\dagger_{kj}$

$$\Rightarrow \hat{\rho}_2 = \sum_{ik} \underbrace{\left(\sum_j U^\dagger_{kj} U_{ji}\right)}_{\delta_{ik}} |\bar{\psi}_i\rangle\langle\bar{\psi}_k|$$

$$\Rightarrow \hat{\rho}_2 = \sum_i |\bar{\psi}_i\rangle\langle\bar{\psi}_i| = \hat{\rho}_1 \quad \checkmark$$

(ii) Now assume $\hat{\rho}_1 = \hat{\rho}_2 \equiv \hat{\rho}$

$\hat{\rho}$ being a Hermitian operator can be diagonalized

$$\Rightarrow \quad \hat{\rho} = \sum_\alpha \lambda_\alpha |e_\alpha\rangle\langle e_\alpha|$$

where $\begin{cases} \sum_\alpha \lambda_\alpha = 1 & \text{with } \lambda_\alpha \text{ real}, \quad 0 \le \lambda_\alpha \le 1 \\ \langle e_\alpha | e_\beta \rangle = \delta_{\alpha\beta} \end{cases}$

Let $|\bar{e}_\alpha\rangle = \sqrt{\lambda_\alpha}\, |e_\alpha\rangle \quad \Rightarrow \quad \hat{\rho} = \sum_\alpha |\bar{e}_\alpha\rangle\langle \bar{e}_\alpha|$

$$\Rightarrow \quad \sum_i |\bar{\Psi}_i\rangle\langle\bar{\Psi}_i| = \sum_\alpha |\bar{e}_\alpha\rangle\langle\bar{e}_\alpha| = \sum_j |\bar{\phi}_j\rangle\langle\bar{\phi}_j|$$

we seek the relationship between $\{|\bar{\Psi}_i\rangle\}$ and $\{|\bar{\phi}_j\rangle\}$

First note $\{|e_\alpha\rangle\}$ form a basis for the Hilbert space (with $\lambda_\alpha = 0$ for those vectors not in $\hat{\rho}$)

$$\Rightarrow \quad |\bar{\Psi}_i\rangle = \sum_\alpha |e_\alpha\rangle\langle e_\alpha|\bar{\Psi}_i\rangle = \sum_\alpha |\bar{e}_\alpha\rangle \frac{\langle e_\alpha|\bar{\Psi}_i\rangle}{\sqrt{\lambda_\alpha}}$$

$$= \sum_\alpha M_{i\alpha}\, |\bar{e}_\alpha\rangle$$

where $M_{i\alpha} = \dfrac{\langle e_\alpha|\bar{\Psi}_i\rangle}{\sqrt{\lambda_\alpha}}$

Now: $\sum_i M_{i\alpha} M^*_{i\beta} = \sum_i \frac{\langle e_\alpha | \Psi_i \rangle \langle \Psi_i | e_\beta \rangle}{\sqrt{\lambda_\alpha \lambda_\beta}}$

$$= \frac{1}{\sqrt{\lambda_\alpha \lambda_\beta}} \langle e_\alpha | \Big(\sum_i |\Psi_i\rangle \langle \Psi_i| \Big) | e_\beta \rangle$$

$$= \frac{1}{\sqrt{\lambda_\alpha \lambda_\beta}} \langle e_\alpha | \hat{\rho} | e_\beta \rangle = \frac{\lambda_\alpha \delta_{\alpha\beta}}{\sqrt{\lambda_\alpha \lambda_\beta}} = \delta_{\alpha\beta}$$

$\Rightarrow$ When arranged in a matrix, the columns of $M_{i\alpha}$ are orthonormal

(Subtle point: $M_{i\alpha}$ need not be square here, since # of pure states in the $\{|\Psi_i\rangle\}$ need not be the dimension of Hilbert space. However, we can always append extra columns in the orthogonal space to make $M_{i\alpha}$ <u>unitary</u>.)

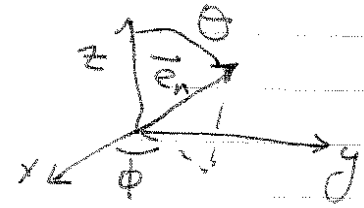
Thus since $|\Psi_i\rangle = \sum_\alpha M_{i\alpha} |\tilde{e}_\alpha\rangle$

$$|\Phi_j\rangle = \sum_\beta N_{j\beta} |\tilde{e}_\beta\rangle$$

$$|\Phi_j\rangle = \sum_i U_{ji} |\Psi_i\rangle$$

where $U = NM^\dagger$ q.e.d.

Problem 3



Given unit direction in space $\mathbb{R}^3$

$$\vec{e}_n = \cos\theta\,\vec{e}_z + \sin\theta(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y)$$

(a)

- We showed in class, spin up along $\vec{e}_n$ is

$$|+_n\rangle = \cos\tfrac{\theta}{2}|+_z\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|-_z\rangle$$

- An arbitrary pure state for two levels

$$|\psi\rangle = \alpha|+_z\rangle + \beta|-_z\rangle \qquad \underline{|\alpha|^2+|\beta|^2 = 1}$$

$$= |\alpha|\,|+_z\rangle + e^{i\delta_{\alpha\beta}}\,|\beta|\,|-_z\rangle$$

where $\delta_{\alpha\beta} = \mathrm{Arg}(\beta) - \mathrm{Arg}(\alpha)$

Thus $|\psi\rangle$ is of the form $|+_n\rangle$ with

$$\theta = 2\cos^{-1}(|\alpha|) \qquad \phi = \delta_{\alpha\beta}$$

q.e.d.

Note: This fact is uniquely true for spin ½

For $J > \frac{1}{2}$, not all pure states can be mapped to directions in space.

(b) Consider a projector onto a pure state

$$\hat{\mathbb{P}}_n = |+_n\rangle\langle +_n|$$

Every operator $\hat{A} = \frac{1}{2}\left(\mathrm{Tr}(\hat{A})\,\hat{\mathbb{1}} + \vec{A}\cdot\hat{\vec{\sigma}}\right)$

where $\vec{A} = \mathrm{Tr}(\hat{A}\,\hat{\vec{\sigma}})$

$$\mathrm{Tr}(\hat{\mathbb{P}}_n) = \langle +_n|+_n\rangle = 1$$

$\mathrm{Tr}(\hat{\mathbb{P}}_n\,\hat{\vec{\sigma}}) = \langle +_n|\hat{\vec{\sigma}}|+_n\rangle = \vec{Q}$ Bloch vector

But for a pure state $\vec{Q} = \vec{e}_n$

$\Rightarrow |+_n\rangle\langle +_n| = \frac{1}{2}\left(\hat{\mathbb{1}} + \vec{e}_n\cdot\hat{\vec{\sigma}}\right)$ q.e.d.

Explicitly $Q_x = \langle +_n|\hat{\sigma}_x|+_n\rangle = \langle +_n|+_z\rangle\langle -_z|+_n\rangle + \text{c.c.}$

$$= \left(\cos\frac{\theta}{2}\right)\left(e^{i\phi}\sin\frac{\theta}{2}\right) + \text{c.c.}$$

$$= \cos\phi\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \cos\phi\sin\theta$$

$$Q_y = \langle +_n|\hat{\sigma}_y|+_n\rangle = \frac{\langle +_n|+_z\rangle\langle -_z|+_n\rangle - \text{c.c.}}{i}$$

$$= \sin\phi\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \sin\phi\sin\theta$$

$$Q_z = \langle +_n|\hat{\sigma}_z|+_n\rangle = |\langle +_n|+_z\rangle|^2 - |\langle +_n|-_z\rangle|^2$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$$

$$\Rightarrow \vec{Q} = \sin\theta(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y) + \cos\theta\,\vec{e}_z$$

$$= \vec{e}_n$$

(c) Consider

$$|\langle +_n | +_{n'}\rangle| = \sqrt{|\langle +_n | +_{n'}\rangle|^2}$$

Now $|\langle +_n | +_{n'}\rangle|^2 = \mathrm{Tr}\left(|+_n\rangle\langle +_n| \; |+_{n'}\rangle\langle +_{n'}| \right)$

$$= \mathrm{Tr}\left[\tfrac{1}{2}(\hat{\mathbb{1}} + \hat{\sigma}_n) \; \tfrac{1}{2}(\mathbb{1} + \hat{\sigma}_{n'}) \right]$$

$$= \frac{1}{4}\underbrace{\mathrm{Tr}(\hat{\mathbb{1}})}_{=2} + \frac{1}{2}\underbrace{\mathrm{Tr}(\hat{\sigma}_n + \hat{\sigma}_{n'})}_{0} + \frac{1}{4}\mathrm{Tr}(\hat{\sigma}_n \hat{\sigma}_{n'})$$

<u>Aside:</u> $\mathrm{Tr}(\hat{\sigma}_n \hat{\sigma}_{n'}) = \mathrm{Tr}(\vec{e}_n \cdot \hat{\vec{\sigma}} \; \vec{e}_{n'} \cdot \vec{\sigma})$

$$= 2\, \vec{e}_n \cdot \vec{e}_{n'}$$

$$\Rightarrow |\langle +_n | +_{n'}\rangle| = \sqrt{\tfrac{1}{2}(1 + \vec{e}_n \cdot \vec{e}_{n'})} = \sqrt{\frac{1+\cos\Theta}{2}}$$

where $\Theta = \cos^{-1}(\vec{e}_n \cdot \vec{e}_{n'})$

$$\Rightarrow \boxed{|\langle +_n | +_{n'}\rangle| = \left|\cos\left(\frac{\Theta}{2}\right)\right|}$$

Anti-podal states on Bloch sphere are orthogonal $\Theta = \pi$

(d) Poincaré sphere

if $|+_z\rangle \Rightarrow$ right-hand circular $= (\vec{e}_x + i\vec{e}_y)/\sqrt{2}$

$|-_z\rangle \Rightarrow$ left-hand circular $= (\vec{e}_x - i\vec{e}_y)/\sqrt{2}$

$$|\pm_x\rangle = \frac{|+_z\rangle \pm |-_z\rangle}{\sqrt{2}} = \begin{cases} \vec{e}_x \\ \vec{e}_y \end{cases} \text{ linear } \begin{matrix} \text{Horizontal} \\ \text{vertical} \end{matrix}$$

$$|\pm_y\rangle = \frac{|+_z\rangle \pm i|-_z\rangle}{\sqrt{2}} = \begin{cases} (\vec{e}_x + \vec{e}_y)/\sqrt{2} \\ (\vec{e}_x - \vec{e}_y)/\sqrt{2} \end{cases} \text{ linear } \begin{matrix} 45^\circ \\ -45^\circ \end{matrix}$$