Physics 566 Quantum Optics
Problem Set #3, Solutions

(ii) Adiabatic rapid passage

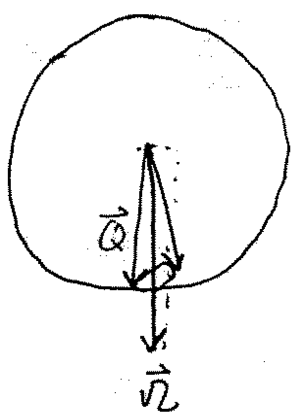
Inhomogenously broadened sample → range of resonance frequencies → range of detunings

Consider one specific "class" of atoms with a given resonance energy. The Hamiltonian in the rotating frame is

$$\vec{H}_{eff} = -\frac{\hbar}{2}\vec{\Omega}_{eff}\cdot\hat{\vec{\sigma}} \quad \text{where } \vec{\Omega}_{eff} = \Omega\vec{e}_x + \Delta\vec{e}_z$$

(a) Atoms start in the ground-state and the laser is detuned well below resonance:

$$|\Delta| \gg \Omega \qquad \Delta < 0$$



$$\Rightarrow \vec{\Omega}_{eff} \approx \Delta\vec{e}_z = -|\Delta|\vec{e}_z$$

The Bloch vector will precess around the pseudo magnetic field at frequency Δ

Now suppose we sweep Δ through resonance slowly compared to $\sqrt{\Omega^2+\Delta^2}$. The spin will then "adiabatically follow" the local field

inhomogenously broadened sample.

(e) Quantitatively, ~~we~~ we can turn to the adiabatic theorem of quantum mechanics

Now with a changing detuning we have the time dependent Hamiltonian in the rotating frame

$$H_{eff} = -\frac{\hbar}{2}\vec{\Omega}_{eff}(t)\cdot\hat{\vec{\sigma}}$$

where $\vec{\Omega}_{eff}(t) = \Omega\vec{e}_x + \Delta(t)\,\vec{e}_z$

We can ~~instantly~~ write down the instaneous eigenvectors and eigenvalues in a snap:

$$\hat{H}_{eff} = -\frac{\hbar}{2}\tilde{\Omega}(t)\,\hat{\sigma}_{n(t)}$$

where $\tilde{\Omega}(t) = \sqrt{\Omega^2 + (\Delta(t))^2}$

and $\hat{\sigma}_{n(t)} = \vec{e}\cdot\hat{\vec{\sigma}}$, $\vec{e}_{n(t)} = \cos\theta(t)\,\vec{e}_z + \sin\theta(t)\,\vec{e}_x$

$$\theta(t) = \tan^{-1}\left(\frac{\Omega}{\Delta(t)}\right)$$

$$\cos\theta(t) = \frac{\Delta(t)}{\tilde{\Omega}(t)} \qquad \sin\theta(t) = \frac{\Omega}{\tilde{\Omega}(t)}$$

We thus have the eigenvalues (since $\hat{\sigma}_n = \pm 1$)

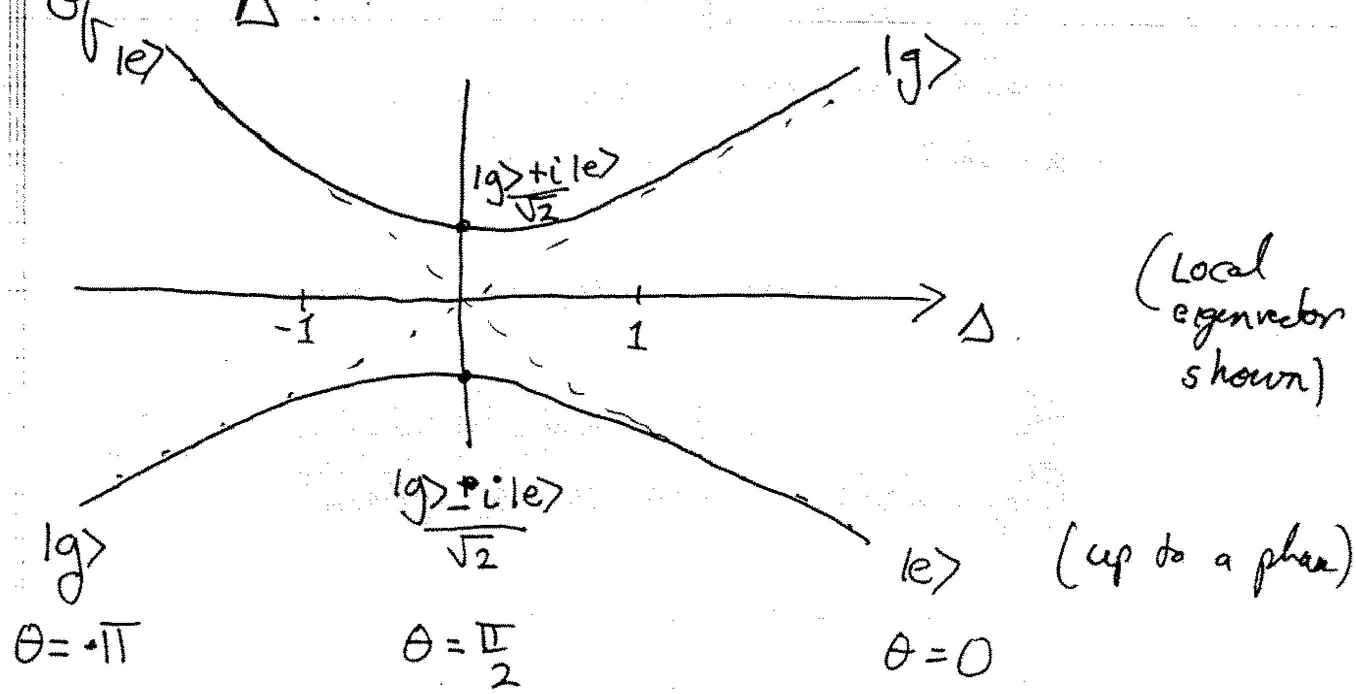
$$E_{\pm}(t) = \mp\frac{\hbar}{2}\tilde{\Omega}(t) = \mp\frac{\hbar}{2}\sqrt{\Omega^2 + \Delta(t)^2}$$

Corresponding to eigenvectors

$$|\pm\rangle_{n(t)} = \cos\frac{\theta(t)}{2}\,|\pm\rangle_z + i\sin\left(\frac{\theta(t)}{2}\right)|\mp\rangle_z$$

$$|+\rangle_z = |e\rangle \qquad |-\rangle_z = |g\rangle$$

A graph of these eigenvalues as a function of Δ:



According to the adiabatic theorem of quantum mechanics, for a time dependent Hamiltonian that varies slowly, if we start in an eigenstate we stay in the local eigenstate. Thus according to the curve above, we we adiabatically follow the lower branch, so that the state evolve from $|g\rangle \rightarrow \frac{1}{\sqrt{2}}(|g\rangle - i|e\rangle) \rightarrow |e\rangle$

This requires $\frac{d}{dt}\theta(t) \ll \tilde{\Omega}(t)$ (adiabatic condition

The local eigenstates of H_{eff} are sometimes known as the "dressed states", since the laser field "dresses" the bare atom states

$$|+\rangle_{n(t)} = \frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{\Delta}{\tilde{\Omega}}}\,|e\rangle + i\sqrt{1-\frac{\Delta}{\tilde{\Omega}}}\,|g\rangle\right)$$

The adiabatic theorem of quantum mechanics says that if we have a Hamiltonian which is time dependent, $\hat{H}(t)$, then given a state at $t=0$ which is an eigenstate of $\hat{H}(0)$

i.e. $|\psi(0)\rangle = |u_n(0)\rangle$ where $\hat{H}(0)|u_n(0)\rangle = E_n(0) |u_n(0)\rangle$

The system will <u>adiabatically follow</u> the Eigenstate (up to a phase)

$$|\psi(t)\rangle \Rightarrow |u_n(t)\rangle$$

if $\hat{H}(t)$ changes <u>slowly</u> compared to the frequency associated with energy splittings.

Here the local eigenstate

$$|\psi(t)\rangle = |+\rangle_{n(t)} = \frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{\Delta}{\tilde{\Omega}}}\,|e\rangle + i\sqrt{1-\frac{\Delta}{\tilde{\Omega}}}\,|g\rangle\right)$$

Well below resonance

$\Delta \ll \Omega$ $\quad |+\rangle_{n(t)} \Rightarrow |g\rangle$

Well above

$\Delta \gg \Omega$ $\quad |+\rangle_{n(t)} \Rightarrow |e\rangle$

Adiabatic if $\left|\frac{1}{\tilde{\Omega}}\frac{d\tilde{\Omega}}{dt}\right| \ll \Omega$

$\uparrow$ smallest splitting

Also require rapid compared to Γ

Problem 2 Light forces on Atom

a) $\hat{H}_{AL}(R) = -\frac{\hbar\Omega(R)}{2}\left(\underbrace{e^{-i\phi(R)}e^{-i\omega_L t}}_{A}|e\rangle\langle g| + \underbrace{e^{+i\phi(R)}e^{i\omega_L t}}_{A^*}|g\rangle\langle e|\right)$

$\Omega(R) = \langle e|\vec{d}\,\hat{\varepsilon}_L|g\rangle E_0(R)$

10

$\hat{F} = -\nabla\hat{H}(R)$

$= +\frac{\hbar}{2}\left[\vec{\nabla}\Omega(R)\right](\quad) + \frac{\hbar}{2}\Omega(R)\,\vec{\nabla}\phi(R)\left(-i\,e^{-i\phi(R)}e^{-i\omega_L t}|e\rangle\langle g| + i\,e^{+i\phi}e^{+i\omega t}|g\rangle\langle e|\right)$

$|\psi\rangle = c_g|g\rangle + c_e|e\rangle$

substituting $\boxed{x = \phi + \omega_L t}$

$\langle\hat{F}\rangle = \langle\psi|F|\psi\rangle = +\frac{\hbar}{2}\left[\vec{\nabla}\Omega(R)\right]\left(c_e^* c_g e^{-i(\phi+\omega_L t)} + c_g^* c_e e^{i(\phi+\omega_L t)}\right)$
$-i\frac{\hbar}{2}\Omega(R)(\vec{\nabla}\phi(R))\left(c_e^* c_g e^{-i(\phi+\omega_L t)} - c_g^* c_e e^{i(\phi+\omega_L t)}\right)$

$= \frac{\hbar}{2}(\vec{\nabla}\Omega(R))\left[\cos x\,(c_e^* c_g + c_g^* c_e) - i\sin x\,(c_e^* c_g - c_g^* c_e)\right]$
$-i\frac{\hbar}{2}\Omega(R)(\vec{\nabla}\phi(R))\left[\cos x\,\underbrace{(c_e^* c_g - c_g^* c_e)}_{2i\,\mathrm{Im}(\rho_{ge})} - i\sin x\,\underbrace{(c_e^* c_g + c_g^* c_e)}_{2\mathrm{Re}(\rho_{ge}) = |u|}\right]$

$= \frac{\hbar}{2}(\vec{\nabla}\Omega(R))\left[2\mathrm{Re}(\rho_{ge})\cos x + 2\mathrm{Im}(\rho_{ge})\sin x\right]$
$+ \frac{\hbar}{2}\Omega\vec{\nabla}\phi(R)\left[2\mathrm{Im}(\rho_{ge})\cos x - 2\mathrm{Re}(\rho_{ge})\sin x\right]$

$= \frac{\hbar}{2}(\vec{\nabla}\Omega(R))\underbrace{\left[2\mathrm{Re}(\rho_{ge}e^{-ix})\right]}_{=u} + \frac{\hbar}{2}\Omega(R)(\vec{\nabla}\phi(R))\underbrace{\left[2\mathrm{Im}\,\rho_{ge}e^{-ix}\right]}_{v}$

$\langle\hat{F}\rangle = \frac{\hbar}{2}(\vec{\nabla}\Omega(R))\,u(t) + \frac{\hbar}{2}\Omega(R)(\vec{\nabla}\phi(R))\,v(t)$ ✓

b) $\langle \frac{dW}{dt} \rangle_{ss} = \langle \vec{v} \cdot \vec{F} \rangle_{period} = \langle \dot{\vec{x}} \cdot q \cdot \vec{E} \rangle_{period} = \langle \dot{\vec{d}} \cdot \vec{E} \rangle_{period}$

$\vec{d}(t) = -e\,\vec{x}(t) = \vec{d}_{eg} \left(e^{-ix} |e\rangle\langle g| + e^{ix} |g\rangle\langle e| \right)$ where $\vec{d}_{eg} = \langle e|\vec{d}|g\rangle$

$\Rightarrow \dot{\vec{d}}(t) = -i\omega_L \vec{d}_{eg} \left(e^{-ix} |e\rangle\langle g| - e^{+ix} |g\rangle\langle e| \right)$

$\Rightarrow \dot{\vec{d}} \cdot \vec{E} = -i\omega_L \hbar \Omega(\vec{R}) \cos x \left(e^{-ix} |e\rangle\langle g| - e^{+ix} |g\rangle\langle e| \right)$

where $\hbar\Omega(R) = \langle e| \vec{d} \cdot \epsilon_L |g\rangle E_0(R)$

$E(\vec{R},t) = \epsilon_L E_0 \cos(\omega_L t + \phi(R))$

$|\psi(t)\rangle = c_g |g\rangle + c_e |e\rangle$

$\Rightarrow \langle \dot{\vec{d}} \cdot \vec{E} \rangle = \langle \psi | \dot{\vec{d}} \vec{E} | \psi \rangle = -i\omega_L \hbar \Omega \cos x \left[c_e^* c_g e^{-ix} - c_g^* c_e e^{+ix} \right]$

$= -i\omega_L \hbar \Omega \cos x \left[\cos x \underbrace{\left(c_e^* c_g - c_g^* c_e \right)}_{2\,\mathrm{Im}(\rho_{ge})} - i \sin x \underbrace{\left(c_e^* c_g + c_g^* c_e \right)}_{2\,\mathrm{Re}(\rho_{ge})} \right]$

$= -i\omega_L \hbar \Omega \left[\underbrace{\cos^2 x}_{\langle \cos^2 x \rangle_{period} = \frac{1}{2}} \; 2\,\mathrm{Im}(\rho_{ge}) - i \underbrace{\left(\tfrac{1}{2} \sin 2x \right)}_{\langle \frac{1}{2} \sin 2x \rangle_{period} = 0} \; 2\,\mathrm{Re}(\rho_{ge}) \right]$

$\langle \dot{\vec{d}} \cdot \vec{E} \rangle_{period} = - i\omega_L \hbar \Omega \cdot \mathrm{Im}(\rho_{ge})$

$\Rightarrow \langle \frac{dW}{dt} \rangle_{ss} = -\hbar \Omega \omega_L \mathrm{Im}(\rho_{ge})_{ss} = -\frac{\hbar}{2} \Omega \omega_L v_{ss}$ ✓

$\langle \frac{dN}{dt} \rangle_{ss} = \frac{1}{\hbar \omega_L} \langle \frac{dW}{dt} \rangle_{ss} = -\frac{\Omega}{2} v_{ss}$ ✓

steady state solution for ρ: $\rho_{ee}^{ss} = -\Omega T_1 \mathrm{Im}(\rho_{ge}^{ss}) = -\frac{\Omega}{\Gamma} \mathrm{Im}(\rho_{ge}) = -\frac{\Omega}{\Gamma} \frac{v_{ss}}{2}$

$\Rightarrow \underbrace{\langle \frac{dN}{dt} \rangle_{ss}}_{\text{rate of absorbed photons}} = \underbrace{\Gamma \rho_{ee}^{ss}}_{\text{rate of sp. emitted photons}}$

The rate of absorbtion is equal to the rate of spontaneous emission.

c) plane wave $\vec{E}(R,t) = \vec{e}_L E_0 \cos(\omega_L t - \vec{k}\vec{R})$ $\qquad \phi(R) = -\vec{k}\vec{R}$

$x = \omega_L t + \phi(R)$

$$F_{diss} = \frac{1}{2}\hbar v(t)\,\Omega(R)\left(\vec{\nabla}\phi(\vec{R})\right)$$

$$= -\hbar\vec{k}\,\frac{v(t)}{2}\,\Omega(R)$$

steady state $v(t) \to v_{ss}$

$$= -\hbar\vec{k}\,\frac{v_{ss}}{2}\,\Omega_0 = \hbar\vec{k}\left\langle\frac{dN}{dt}\right\rangle_{ss}$$

with $\left\langle\frac{dN}{dt}\right\rangle = -\frac{\Omega}{2}v_{ss}$

$$\Rightarrow \underline{F_{diss} = \hbar k\left\langle\frac{dN}{dt}\right\rangle_{ss}} \checkmark$$

$\hbar k$ momentum of one photon

$N\hbar k$ total momentum

$\Bigg\} \Rightarrow F = +\left\langle\frac{dp}{dt}\right\rangle$

$\Rightarrow$ F_{diss} is the force due to radiation pressure

d) $F_{react} = \frac{\hbar}{2}u(t)\vec{\nabla}\Omega(\vec{R})$ $\longrightarrow$ steady state

$$\dot{u} = 2\,\mathrm{Re}(\dot{\rho}_{ge}) = -\frac{1}{T_2}u + \Delta v \overset{!}{=} 0 \qquad u_{ss} = T_2\Delta v_{ss}$$

$$\dot{v} = 2\,\mathrm{Im}(\dot{\rho}_{ge}) = -\frac{1}{T_2}v - \Delta u + \Omega w \overset{!}{=} 0 \qquad v_{ss} = T_2(\Omega w_{ss} - \Delta u_{ss})$$

$$\dot{w} = 2\,\dot{\rho}_{ee} = -\frac{1}{T_1}(w+1) - \Omega v \overset{!}{=} 0 \qquad w_{ss} = -T_1\Omega v_{ss} - 1$$

using $s = \dfrac{\Omega^2/2}{\Delta^2 + \frac{\Gamma^2}{4}}$ we get for the steady state equations

$$u_{ss} = \frac{2\Delta}{\Omega}\,\frac{s}{1+s}$$

$$v_{ss} = \frac{2\Gamma}{2\Omega}\,\frac{s}{1+s}$$

$$w_{ss} = \frac{-1}{2(1+s)}$$

For weak saturation $s \ll 1$

$\rightarrow u_{ss} = \frac{2\Delta s}{\Omega}$

$\Rightarrow F_{react} = \frac{\hbar}{2} \frac{2\Delta s}{\Omega} \vec{\nabla}\Omega = \hbar \frac{\Delta s}{\Omega} \vec{\nabla}\Omega$

Using $s = \frac{\Omega^2(R)/2}{\Delta^2 + \frac{\Gamma^2}{4}}$ and $\vec{\nabla} s = 2(\vec{\nabla}\Omega(R)) \frac{\Omega(R)}{2} \frac{1}{\Delta^2 + \frac{\Gamma^2}{4}} = 2\frac{s}{\Omega}(\vec{\nabla}\Omega)$

$\Rightarrow F_{react} = \frac{\hbar\Delta}{2} \vec{\nabla} s = -\vec{\nabla}(U(R))$ ✓

$\langle \vec{d} \cdot E \rangle = \hbar\Omega(R) \cos x \left[\cos x \left(c_e^* c_g + c_g^* c_e\right) - i \sin x \left(c_e^* c_g - c_g^* c_e\right)\right]$

$= \hbar\Omega(R) \left[\underbrace{\cos^2 x}_{=\frac{1}{2}} \, 2\,\mathrm{Re}(\rho_{ge}) - i \underbrace{\sin x \cos x}_{=0} \, 2\,\mathrm{Im}(\rho_{ge})\right]$

time average $\rightarrow$

$= \hbar\Omega(R) \mathrm{Re}(\rho_{ge}) = \frac{\hbar}{2}\Omega(R)\, u_{ss} = \frac{\hbar}{2}\Omega(R) \frac{2\Delta}{\Omega} \underbrace{\frac{s}{1+s}}_{\approx s}$

$= \hbar\Delta s$

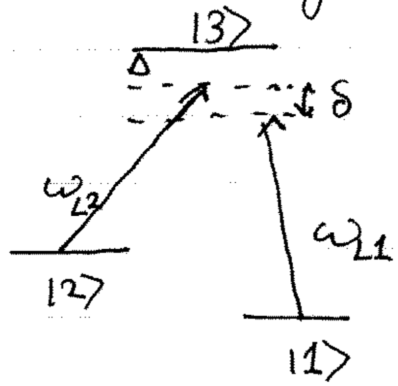
$U(R) = -\frac{1}{2}\langle \vec{d} \cdot \vec{E} \rangle = -\frac{1}{2}\hbar\Delta s$ ✓ qed

Physics 566 - Quantum Optics

Problem Set #3 Solutions

Problem 3: Raman-Rabi Flopping

3-level system driven by two fields



$$\Delta = \omega_{32} - \omega_{L2}$$

$$\delta = \omega_{12} - \Delta\omega_L$$

where $\omega_{ij} \equiv \frac{E_i - E_j}{\hbar}$

$\Delta\omega_L \equiv \omega_{L2} - \omega_{L1}$

(a) Hamiltonian: $\hat{H} = \hat{H}_A + \hat{H}_{AL}$

$$\hat{H}_A = \sum_{j=1}^{3} E_j |j\rangle\langle j| \, ; \qquad \hat{H}_{AL} \overset{RWA}{=} -\frac{\hbar\Omega_1}{2}\left(e^{-i\omega_{L1}t}|3\rangle\langle 1| + H.c.\right) - \frac{\hbar\Omega_2}{2}\left(e^{-i\omega_{L2}t}|3\rangle\langle 2| + H.c.\right)$$

Define: $\left\{ \begin{array}{l} |\tilde{\psi}\rangle = \hat{U}^\dagger |\psi\rangle, \quad \tilde{H} = \hat{U}^\dagger \hat{H} \hat{U} + i\hbar \frac{\partial \hat{U}^\dagger}{\partial t} \hat{U} \\ \hat{U} = \sum_{j=1}^{1} e^{-i\lambda_j t} |j\rangle\langle j| \end{array} \right.$

$$\Rightarrow \tilde{H}_A = \sum_{j=1}^{3} E_j \underbrace{\hat{U}^\dagger |j\rangle\langle j| \hat{U}^\dagger}_{|j\rangle\langle j|} - \hbar \sum_{j} \lambda_j |j\rangle\langle j| = \sum_j (E_j - \hbar\lambda_j)|j\rangle\langle j|$$

$$\tilde{H}_{AL} = -\frac{\hbar\Omega_1}{2}\left(e^{-i(\omega_{L1} - (\lambda_3 - \lambda_1))t} |3\rangle\langle 1| + H.c.\right) - \frac{\hbar\Omega_2}{2}\left(e^{-i(\omega_{L2} - (\lambda_3 - \lambda_2))t} |3\rangle\langle 2| + H.c.\right)$$

Thus we see that the effect of the unitary is to shift the eigenvalue $E_j \Rightarrow E_j \mp \hbar\lambda_j$. We can thus remove the explicit time dependence in the Hamiltonian by choosing

$$\lambda_3 - \lambda_1 = \omega_{L1} \qquad \lambda_3 - \lambda_2 = \omega_{L2}$$

These are two equations for three unknowns. This means that the absolute zero of energy is at our disposal

Choosing $\lambda_2 = +E_2/\hbar$ shifts level $|2\rangle \Rightarrow$ zero energy

$$\Rightarrow \lambda_3 = +\frac{E_2}{\hbar} + \omega_{L2} \qquad \lambda_1 = \lambda_3 - \omega_{L1} = -\frac{E_2}{\hbar} + \omega_{L2} - \omega_{L1}$$

$$\therefore \tilde{H} = \left(\frac{E_1 - E_2}{\hbar} + \omega_{L2} - \omega_{L1}\right)|1\rangle\langle 1| + \left(\frac{E_3 - E_2}{\hbar} - \omega_{L2}\right)|3\rangle\langle 3|$$
$$- \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) - \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right)$$

$$\Rightarrow \boxed{\tilde{H} = \hbar\delta|1\rangle\langle 1| - \hbar\Delta|3\rangle\langle 3| - \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) - \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right)}$$

With δ and Δ defined on page 1

(b) Adding decay

$$\frac{d\tilde\rho}{dt} = \frac{1}{i\hbar}[\tilde H, \tilde\rho] + \mathcal{L}_{relax}[\tilde\rho]$$

$$\mathcal{L}_{relax}[\tilde\rho] = -\frac{\Gamma_{31}}{2}\left(|3\rangle\langle 3|\tilde\rho + \tilde\rho|3\rangle\langle 3| - 2|1\rangle\langle 3|\tilde\rho|3\rangle\langle 1|\right)$$
$$-\frac{\Gamma_{32}}{2}\left(|3\rangle\langle 3|\tilde\rho + \tilde\rho|3\rangle\langle 3| - 2\,|2\rangle\langle 3|\tilde\rho|3\rangle\langle 2|\right)$$

$$\Rightarrow \dot{\tilde\rho}_{ij} = \frac{1}{i\hbar}\left(\langle i|\tilde H\tilde\rho|j\rangle - \langle i|\tilde\rho\tilde H|j\rangle\right) + \langle i|\mathcal{L}_{relax}[\tilde\rho]|j\rangle$$

I will show explicit evaluation of one of the matrix ~~elemenar~~ elements. The others follow in the same manner.

E.g.
$$\dot\rho_{23} = \frac{1}{i\hbar}\langle 2|\tilde H_A\tilde\rho|3\rangle - \frac{1}{i\hbar}\langle 2|\tilde\rho\tilde H_A|3\rangle$$
$$\frac{1}{i\hbar}\langle 2|\tilde H_{AL}\tilde\rho|3\rangle - \frac{1}{i\hbar}\langle 2|\tilde\rho\tilde H_{AL}|3\rangle$$
$$+\langle 3|\mathcal{L}_{relax}[\tilde\rho]|3\rangle$$

Plug in $\tilde H$:

$$\langle 2|\tilde H_A\tilde\rho|3\rangle = 0 \qquad \langle 2|\tilde\rho\tilde H_A|3\rangle = \hbar\Delta\tilde\rho_{23}$$

$$\langle 2|\tilde H_{AL}\tilde\rho|3\rangle = -\frac{\hbar\Omega_2}{2}\tilde\rho_{33} \qquad \langle 2|\tilde\rho\tilde H_{AL}|3\rangle = -\frac{\hbar\Omega_2}{2}\tilde\rho_{22} - \frac{\hbar\Omega_1}{2}\tilde\rho_{21}$$

$$\langle 2|\mathcal{L}_{relax}[\tilde\rho]|3\rangle = -\left(\frac{\Gamma_{31}}{2} + \frac{\Gamma_{32}}{2}\right)\tilde\rho_{23} = -\frac{\Gamma_3}{2}\tilde\rho_{23}$$

$$\Rightarrow \dot{\rho}_{23} = \left(-i\Delta - \frac{\Gamma_3}{2}\right)\rho_{23} + i\frac{\Omega_2}{2}(\rho_{33}-\rho_{22}) - i\frac{\Omega_2}{2}\rho_{12}$$

Note: there was error in the assignment ($\Gamma_{01} \to \Gamma_3$ in this eq.)

The other equations follow similarly. Note that the laser induced coherence between levels $|2\rangle$ and $|3\rangle$ and $|1\rangle$ and $|3\rangle$, ie, ρ_{23}, ρ_{13}, act as source terms for ρ_{12}, coherence between $|1\rangle$ and $|2\rangle$

(b') In the limit $\Delta \gg \Omega, \Gamma, \delta$ we can adiabatically eliminate ρ_{33} and coherence ρ_{23} ρ_{12}.

Set these time derivatives to zero for these "fast" variables (which are slaves to the slow

$$\Rightarrow \rho_{33} = -\frac{\Omega_1}{\Gamma_3}\frac{(\rho_{13}-\rho_{31})}{2i} - \frac{\Omega_2}{\Gamma_3}\frac{(\rho_{23}-\rho_{32})}{2i}$$

$$\Rightarrow \rho_{33} = -\frac{\Omega_1}{\Gamma}\,\mathrm{Im}(\rho_{13}) - \frac{\Omega_2}{\Gamma_3}\,\mathrm{Im}(\rho_{23})$$

$$\rho_{13} = \frac{1}{2\Delta - i\Gamma_3}\left(\Omega_1(\rho_{33}-\rho_{11}) - \Omega_2\rho_{12}\right)$$

$$\rho_{23} = \frac{1}{2\Delta - i\Gamma_3}\left(\Omega_2(\rho_{33}-\rho_{11}) - \Omega_1\rho_{21}\right)$$

The remaining equations are for the "slow variables" ρ_{11}, ρ_{22}, ρ_{12}

(c) To lowest order in $\frac{\Omega_1}{\Gamma}$ and $\frac{\Omega}{\Delta}$ we can neglect the contribution of ρ_{33} to ρ_{23} and ρ_{13}

$$\Rightarrow \rho_{13}^{(0)} \cong \frac{1}{2\Delta \cancel{- i\Gamma_3}}\left(-\Omega_1 (\rho_{11}) - \Omega_2 \rho_{12}\right)$$

$$\rho_{23}^{(0)} \cong \frac{1}{2\Delta \cancel{- i\Gamma_3}}\left(-\Omega_2 (\rho_{22}) - \Omega_1 \rho_{21}\right)$$

neglecting for $\Delta \gg \Gamma_3$

Since these are all first order in Ω

$$\Rightarrow \rho_{33}^{(0)} \cong 0 \text{ to first order in } \frac{\Omega}{\Gamma}, \frac{\Omega}{\Delta}$$

Plugging $\rho_{13}^{(0)}$ and $\rho_{23}^{(0)}$ back into equation for ρ_{12} etc

$$\Rightarrow \dot{\rho}_{11} = \cancelto{0}{\Gamma_{31}\rho_{33}^{(0)}} + \Omega_2 \operatorname{Im}(\rho_{13}^{(0)})$$

$$= -\frac{\Omega_2\Omega_1}{2\Delta}\cancelto{0}{\operatorname{Im}(\rho_{22})} + \frac{\Omega_2\Omega_1}{2\Delta}\operatorname{Im}(\rho_{21})$$

$$= -i\frac{\Omega_2\Omega_1}{4\Delta}(\rho_{21} - \rho_{12}) = -i\frac{\Omega_{eff}}{2}(\rho_{21} - \rho_{12})$$

where $\Omega_{eff} = \frac{\Omega_2\Omega_1}{2\Delta}$

$$\dot{\rho}_{22} = \cancelto{0}{\Gamma_{32}\rho_{33}^{(0)}} + \Omega_2 \operatorname{Im}(\rho_{23}^{(0)})$$

$$= \frac{\Omega_2\Omega_1}{2\Delta}\operatorname{Im}(\rho_{12}) = -\dot{\rho}_{11}$$

(Since $\operatorname{Im}\rho_{12} = -\operatorname{Im}\rho_{21}$

Finally:

$$\dot{\rho}_{12} = -i\delta\,\rho_{12} + i\frac{\Omega_1}{2}\rho_{32}^{(0)} - i\frac{\Omega_2}{2}\rho_{13}^{(0)}$$

$$= -i\delta\,\rho_{12} + i\frac{\Omega_1}{2}\left(-\frac{\Omega_1}{2\Delta}\rho_{22} - \frac{\Omega_2}{2\Delta}\rho_{12}\right) - i\frac{\Omega_2}{2}\left(-\frac{\Omega_2}{2\Delta}\rho_{11} - \frac{\Omega_1}{2\Delta}\rho_{12}\right)$$

$$= -i\delta\,\rho_{12} + i\frac{\Omega_{eff}}{2}\left(\rho_{11} - \rho_{22}\right)$$

We thus arrive at an effective 2-level system

$$\boxed{\begin{aligned} \dot{\rho}_{11} &= -\dot{\rho}_{22} = -i\frac{\Omega_{eff}}{2}\left(\rho_{21} - \rho_{12}\right) \\ \dot{\rho}_{12} &= -i\Delta_{eff}\,\rho_{12} - i\frac{\Omega_{eff}}{2}\left(\rho_{22} - \rho_{11}\right) \\ \text{where}\quad \Omega_{eff} &= \frac{\Omega_1\Omega_2}{2\Delta} \qquad \Delta_{eff} = \delta = \omega_{12} - \Delta\omega_L \end{aligned}}$$

Raman-detuning

Note: In general there will be an AC-Stark contribution of Δ_{eff} which we have neglected by set $\rho_{33} \to 0$

(d) Had we kept higher order terms, we would have found the decay rate for our effective two-level system:

$$\gamma_{eff} = \left(\frac{S_1 + S_2}{2}\right)\Gamma_3 \qquad \text{where} \quad S_{1,2} = \frac{2\Omega_{1,2}^2}{4\Delta_{1,2}^2 + \Gamma_3^2}$$

are the saturation parameters for transitions 1,2

$$\Rightarrow \quad \gamma_{eff} = \left(\frac{\Omega_1^2+\Omega_2^2}{4\Delta^2+\Gamma_3^2}\right)\Gamma_3 \approx \frac{\Omega^2}{\cancel{4}\Delta}\left(\frac{\Gamma_3}{\Delta}\right)$$

Having assumed $\Delta \gg \Gamma_3$ and $\Delta_1 \approx \Delta_2$, $\Omega_1 \approx \Omega_2$,

we have $\Omega_{eff} = \frac{\Omega_1 \Omega_2}{2\Delta} \approx \frac{\Omega^2}{2\Delta}$

Coherent Rabi flopping $\Rightarrow$ $\Omega_{eff} \gg \gamma_{eff}$

$$\Rightarrow \boxed{\Delta \gg \Gamma_3 \quad \text{as we have assumed!}}$$

Coherent Rabi flopping on a Raman transition is much easier to achieve than on a strong 2-level resonance because linewidth of the transition can be reduced. Furthermore effective laser linewidth is very narrow since it depends on the difference frequency. Thus if the two beams can be obtained from the same laser, through some kind of modulator, the effective linewidth is very narrow.

Most coherent control and manipulation with atoms employs Raman resonances.