Physics 566, Quantum Optics

Problem Set #4

Solutions

1. Free induction decay : recall the OBEs for a 2 level atom

with $\vec{R} = U e_x + V e_y + W e_z$; $\vec{Q} = \Omega e_x + \Delta e_z$

$\dot{\vec{R}} = \vec{R} \times \vec{Q}$, which is the same as: $\frac{d}{dt}\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 0 & \Delta & 0 \\ -\Delta & 0 & \Omega \\ 0 & -\Omega & 0 \end{pmatrix}\begin{pmatrix} U \\ V \\ W \end{pmatrix}$

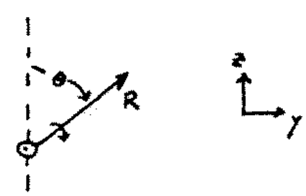
adding the (non-Hamiltonian) decay terms:

$$\begin{aligned} \dot{U} &= \Delta V - \Gamma/2\, U \\ \dot{V} &= -\Delta U + \Omega W - \Gamma/2\, V \\ \dot{W} &= -\Omega V - \Gamma/2 - \Gamma W \end{aligned}$$

For this part, we have $\Gamma \to 0$ and $\Delta = 0$, so:

$$\begin{aligned} \dot{U} &= 0 \\ \dot{V} &= \Omega W \\ \dot{W} &= -\Omega V \end{aligned} \qquad \text{with } W(t=0) = -\tfrac{1}{2} \quad \text{(atom in gnd. state)}$$

It is useful to keep the Bloch vector picture in mind while doing this problem:



The $\vec{Q}$ vector is out of the paper (along x) for $\Delta = 0$, and $\vec{R}$ rotates as shown, with an instantaneous angular velocity $\dot{\theta} = \Omega$

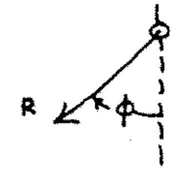
This can be derived directly from the Bloch equations:

$$\tan\theta = \frac{V}{W}$$

$$(1+\tan^2\theta)\,\dot{\theta} = \frac{\dot{V}}{W} - \frac{V}{W^2}\dot{W} = \Omega + \frac{\Omega V^2}{W^2} = (1+\tan^2\theta)\,\Omega$$

$$\dot{\theta} = \Omega$$

Now take ϕ to be the angle of $\vec{R}$ with respect to its initial position (the gnd state) along $-z$: we have $\dot{\phi} = \dot{\theta}$



To have $\rho_{ee} = 1/2$, we want $\phi = \pi/2$

$$\phi = \int_0^T \dot{\phi}\, dt = \int_0^T \Omega\, dt = \pi/2$$

now $\Omega(t) = \dfrac{\vec{E}(t)\cdot\vec{d}}{\hbar}$ so, assuming $\vec{d}$ along $\vec{E}$ we have.

$$\boxed{\int_0^T E(t)\, dt = \frac{\pi}{2}\frac{\hbar}{d}}$$

The initial condition was $W = 1/2$, $U = V = 0$, and the length of $\vec{R}$ does not change, so at $\phi = \pi/2$, $V = -1/2$

since $V = \mathrm{Im}\, \tilde{\rho}_{eg} = \mathrm{Im}\, \rho_{eg} e^{i\omega t}$, we have.

$$\boxed{\tilde{\rho}_{eg} = \frac{-i}{2} \qquad \rho_{eg} = \frac{-i}{2} e^{-i\omega t}}$$

1b. To avoid problems with system of units (cgs vs. S.I.) we will write unit independent expressions

For a constant amplitude $\pi/2$ pulse $\dfrac{\vec{E}\cdot\vec{d}}{\hbar} T = \pi/2$, or for $\vec{E}$ along $\vec{d}$

$$E = \frac{\pi\hbar}{2dT}$$. Writing this in terms of the radial matrix element x:

$$E^2 = \frac{\pi^2\hbar^2}{4T^2e^2x^2}.$$

Now we use the unit-independent expression for the decay rate Γ:

$$\Gamma = \frac{4}{3}\alpha\frac{\omega_0^3}{c^2}x^2 \qquad \alpha = \text{fine structure constant}$$

or $$\frac{1}{x^2} = \frac{4\alpha}{3\Gamma}\,\frac{(2\pi)^3 c}{\lambda^3}.$$

So far everything is good in either SI or cgs, but the expression for the intensity I does depend on units:

$$I = \frac{c}{2}\frac{[4\pi\epsilon_0]}{4\pi} E^2$$ where the term in [ ] is used for SI units

inserting the expression for E^2 and arranging terms we have

$$I = \frac{1}{8\pi}\left\{\frac{[4\pi\epsilon_0]\hbar c}{e^2}\right\}\frac{\hbar\pi^2}{4T^2 x^2}$$ but $$\frac{e^2}{\hbar c[4\pi\epsilon_0]} = \alpha$$

so, inserting the expression for $1/x^2$:

$$I = \frac{\hbar\pi}{32T^2\alpha}\,\frac{4}{3}\,\frac{\alpha}{\Gamma}\,\frac{(2\pi)^3 c}{\lambda^3} = \boxed{\frac{\pi^4\hbar c}{3\Gamma\lambda^3 T^2} = I}$$

this is good in either cgs or SI. Evaluating in SI units

$$I = \frac{\pi^4\cdot 1.05\times10^{-34}\,J\cdot s\cdot 3\times10^8\,m/s\cdot 16\times10^{-9}\,s}{3\cdot(10^{-10}\,s)^2\,(0.589\times10^{-6}\,m)^3} = 8\times10^6\,\frac{W}{m^2}$$

$$\boxed{I = 800\,\frac{W}{cm^2}}$$ for a 100 ps $\pi/2$ pul.

1c. Compare $\Omega = \Gamma$ (part c) with $\Omega = \frac{\pi}{2T}$ (part b)

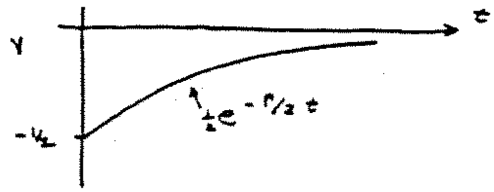
$$\frac{I_c}{I_b} = \frac{\Omega_c^2}{\Omega_b^2} = \left(\frac{2T\Gamma}{\pi}\right)^2 = \left(\frac{2T}{\pi\,\tau_{nat}}\right)^2 = \left(\frac{2\times10^{-10}\,sec}{\pi\cdot16\times10^{-9}\,sec}\right)^2 = 1.58\times10^{-5}$$

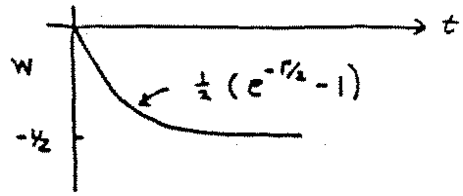
$$\boxed{\frac{I_c}{I_b} = 1.58\times10^{-5}}$$

$$I_b = 1.58\times10^{-5}\cdot 800\,W/cm^2 = \boxed{12.7\,\frac{mW}{cm^2}}$$ for $\Omega = \Gamma$

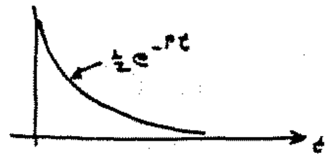
with $\Omega = 0$, $\Delta = 0$ the OBEs are:

$$\dot U = -\Gamma/2\, U$$
$$\dot V = -\Gamma/2\, V$$
$$\dot W = -\Gamma/2 - \Gamma W$$

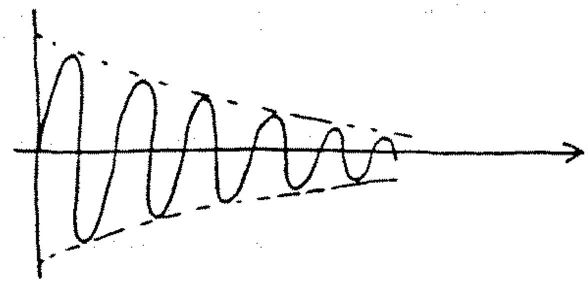




$\rho_{ee} = W + 1/2$



$\rho_{eg} = -iVe^{-i\omega t}$ $\quad$ $\mathrm{Re}\,\rho_{eg} = \frac{1}{2}\sin\omega t\, e^{-\Gamma/2\, t}$



2a The OBEs in steady state:

$$\dot U = \Delta V - \Gamma/2\, U = 0$$

$$\dot V = -\Delta U + \Omega W - \Gamma/2\, V = 0$$

$$\dot W = -\Omega V - \Gamma/2 - \Gamma W = 0$$

solving these simultaneously leads to:

$$W = -\tfrac{1}{2}\left(\frac{2\Delta^2 + \Gamma^2/2}{\Omega^2 + \Gamma^2/2 + 2\Delta^2}\right) \qquad \boxed{\rho_{ee} = W + 1/2 = \frac{\Omega^2/\Gamma^2}{1 + 2\Omega^2/\Gamma^2 + 4\Delta^2/\Gamma^2}}$$

$$V = \frac{-\Gamma}{\Omega}(W + 1/2) = \boxed{\frac{-\Omega/\Gamma}{1 + 2\Omega^2/\Gamma^2 + 4\Delta^2/\Gamma^2} = V} \qquad U = \frac{2\Delta}{\Gamma} V = \boxed{\frac{-2\Delta\Omega/\Gamma^2}{1 + 2\Omega^2/\Gamma^2 + 4\Delta^2/\Gamma^2} = U}$$

(note that U, V, W have the same denominator)

Problem 4 Dark states

a)

see problem 3 : $\hat{H}$ in rotating Frame and RWA

$$\hat{H} = \hbar\delta\, |1\rangle\langle 1| - \hbar\Delta\, |3\rangle\langle 3| - \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) - \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right)$$

here: $\delta = 0$, $\Delta = 0$ (on resonance)

$|1\rangle \hat{=} |g_1\rangle$

$|2\rangle \hat{=} |g_2\rangle$

$|3\rangle \hat{=} |e\rangle$

$$\Rightarrow \hat{H} = -\frac{\hbar}{2}\Omega_1\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) - \frac{\hbar}{2}\Omega_2\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)$$

$$\hat{H} = -\frac{\hbar}{2}\begin{array}{c} \begin{array}{ccc} |g_1\rangle & |g_2\rangle & |e\rangle \end{array} \\ \begin{pmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{pmatrix} \end{array} \begin{array}{c} |g_1\rangle \\ |g_2\rangle \\ |e\rangle \end{array}$$

eigenvalues and eigenvectors

eigenvalue $|\lambda I - H| \stackrel{!}{=} 0$

$$\begin{vmatrix} \lambda & 0 & \frac{\hbar\Omega_1}{2} \\ 0 & \lambda & \frac{\hbar\Omega_2}{2} \\ \frac{\hbar\Omega_1}{2} & \frac{\hbar\Omega_2}{2} & \lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^3 - \lambda\left(\frac{\hbar\Omega_1}{2}\right)^2 - \lambda\left(\frac{\hbar\Omega_2}{2}\right)^2$$

$$\underline{\lambda_{1,2} = \pm\frac{\hbar}{2}\sqrt{\Omega_1^2 + \Omega_2^2}}$$

$$\underline{\lambda_3 = 0}$$

eigenvectors

3) $\begin{pmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = 0 \quad \Rightarrow \quad |\psi_3'\rangle = \dfrac{\Omega_2 |g_1\rangle - \Omega_1 |g_2\rangle}{\sqrt{\Omega_1^2+\Omega_2^2}}$

2) $-\frac{\hbar}{2} \begin{pmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = -\frac{\hbar}{2} \sqrt{\Omega_1^2+\Omega_2^2} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \sqrt{\Omega_1^2+\Omega_2^2} \end{pmatrix}$

$\Rightarrow \quad |\psi_2'\rangle = \Omega_1 |g_1\rangle + \Omega_2 |g_2\rangle + \sqrt{\Omega_1^2+\Omega_2^2}\, |e\rangle$

1) $-\frac{\hbar}{2} \begin{pmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = +\frac{\hbar}{2} \sqrt{\Omega_1^2+\Omega_2^2} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -\Omega_1 \\ -\Omega_2 \\ \sqrt{\Omega_1^2+\Omega_2^2} \end{pmatrix}$

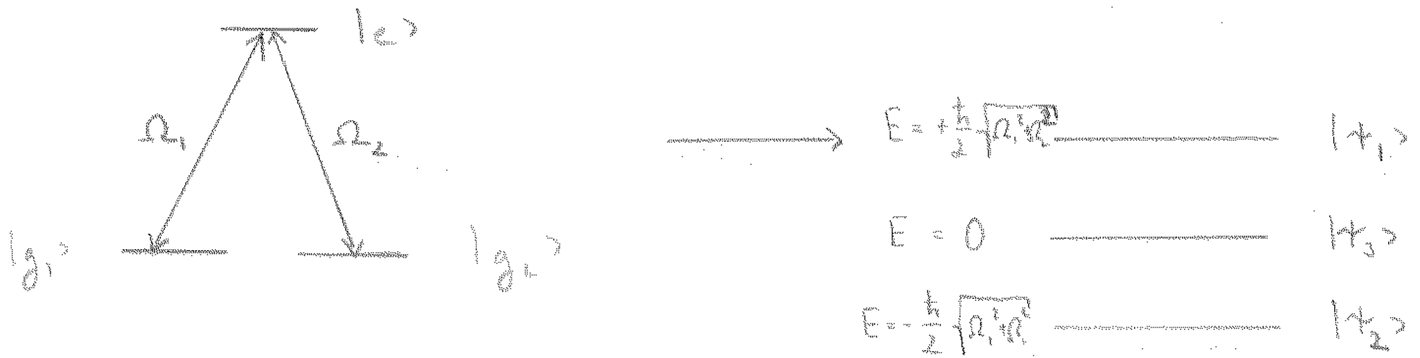
$\Rightarrow \quad |\psi_1'\rangle = -\Omega_1 |g_1\rangle - \Omega_2 |g_2\rangle + \sqrt{\Omega_1^2+\Omega_2^2}\, |e\rangle$

normalized eigen states

$\Rightarrow \quad |\psi_{1,2}\rangle = \frac{1}{\sqrt{2}} \left(\mp \frac{\Omega_1}{\sqrt{\Omega_1^2+\Omega_2^2}} \mp \frac{\Omega_2}{\sqrt{\Omega_1^2+\Omega_2^2}} + 1 \right) \qquad E_{1,2} = \pm \frac{\hbar}{2} \sqrt{\Omega_1^2+\Omega_2^2}$

$|\psi_3\rangle = \frac{1}{\sqrt{\Omega_1^2+\Omega_2^2}} \left(\Omega_2 |g_1\rangle - \Omega_1 |g_2\rangle \right) \qquad E_3 = 0$

dressed states



Why is the "antisymmetric" state $|\psi_2\rangle = \Omega_2 |g_1\rangle - \Omega_1 |g_2\rangle$ a dark state?

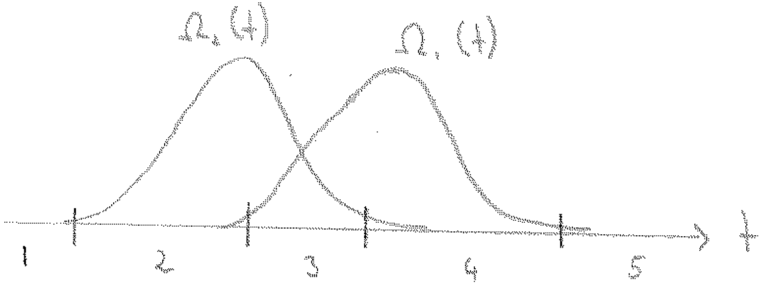
One can think about this as an interference between the two different transitions $|g_1\rangle \to |e\rangle$ and $|g_2\rangle \to |e\rangle$. Depending on the phase ϕ $\left(|\psi\rangle = \Omega_2 |g_1\rangle + e^{i\phi} \Omega_1 |g_2\rangle\right)$ and the strength of interaction (Ω_1 and Ω_2) we get total destructive interference.

$\Longrightarrow$ $|\psi\rangle = \Omega_2 |g_1\rangle - \Omega_1 |g_2\rangle$ does not couple to the excited state and is therefore a dark state

Note: The whole Lamba system is effectively reduced to a two level system where the excited state is coupled only to the "symmetric" state $|\psi\rangle = \Omega_2 |g_1\rangle + \Omega_1 |g_2\rangle$.

b) adiabatic transfer through non intuitive pulse sequence

we can take a look at the eigenstates of the system in each time intervall



1) uncoupled states

$|e\rangle$

$|g_1\rangle$ $|g_2\rangle$

all population in $|g_1\rangle$

2) pulse $\Omega_2(t)$

$|\phi_-\rangle = |g_2\rangle - |e\rangle$

$+\frac{\hbar}{2}\Omega_2$

$|g_1\rangle$

$-\frac{\hbar}{2}\Omega_2$

$|\phi_+\rangle = |g_2\rangle + |e\rangle$

all population still in $|g_1\rangle$

population transfer to $|g_2\rangle$ with out transfer to the excited state

3) both pulses overlap ⟶ see part a)

$-|g_1\rangle - |g_2\rangle + \Gamma\, |e\rangle$

$\sqrt{\Omega_1^2 + \Omega_2^2}$

$|g_1\rangle - |g_2\rangle$

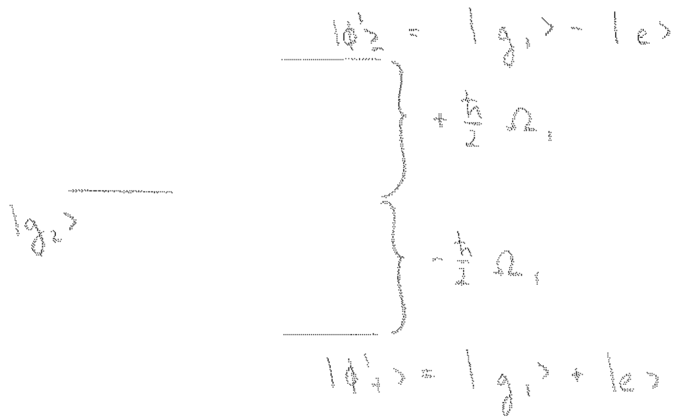
$\sqrt{\Omega_1^2 + \Omega_2^2}$

$+|g_1\rangle + |g_2\rangle + \Gamma\, |e\rangle$

population in state

$|+\rangle = \Omega_2 |g_1\rangle - \Omega_1 |g_2\rangle$

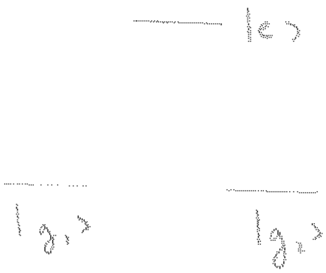
4) pulse $\Omega_1(t)$



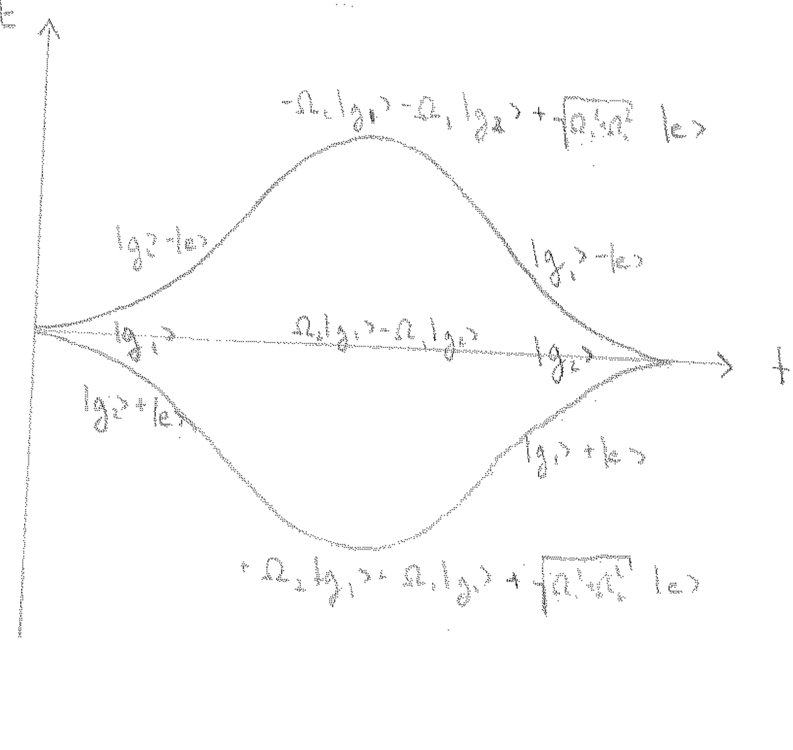
population in $|g_2\rangle$

population in $|g_2\rangle$

5) no coupling



$\longrightarrow$ We can now draw the dressed state eigenvalues as a funtion of time



The population stays in the state with zero eigenvalue, i.e. in the local dark state.

$\Longrightarrow$ adiabatic population transfer from $|g_1\rangle$ to $|g_2\rangle$

Problem 3: Momentum and Angular Momentum in Field

From Maxwell's Equation

$$\vec{P} = \int d^3\vec{x} \; \frac{\vec{E}(\vec{x}) \times \vec{B}(\vec{x})}{4\pi c} = \vec{p}(\vec{x}) \quad \text{momentum density}$$

$$\vec{J} = \int d^3\vec{x} \, \left(\vec{x} \times \vec{p}(\vec{x}) \right)$$

Quantized field $\hat{\vec{A}}(\vec{x}) = \hat{\vec{A}}^{(+)}(\vec{x}) + \hat{\vec{A}}^{(-)}(\vec{x})$

$$\hat{\vec{A}}^{(+)}(\vec{x}) = \sum_{\vec{k},\lambda}{}' \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \, \vec{\epsilon}_{\vec{k},\lambda} \, e^{i\vec{k}\cdot\vec{x}} \, \hat{a}_{\vec{k},\lambda}$$

$$\hat{\vec{A}}^{(-)} = \left(\hat{\vec{A}}^{(+)}\right)^{\dagger}$$

$$\hat{\vec{E}}^{(+)} = i \sum_{\vec{k},\lambda}{}' \sqrt{\frac{2\pi\hbar\omega_k}{V}} \, \vec{\epsilon}_{\vec{k},\lambda} \, e^{i\vec{k}\cdot\vec{x}} \, \hat{a}_{\vec{k},\lambda}$$

$$\hat{\vec{B}}^{(+)} = i \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar\omega_k}{V}} \left(\vec{e}_k \times \vec{\epsilon}_{\vec{k},\lambda}\right) e^{i\vec{k}\cdot\vec{x}} \, \hat{a}_{\vec{k},\lambda}$$

Notes: $\displaystyle\int_V d^3x \, \frac{e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}{V} = \delta_{\vec{k},\vec{k}'}$

$$\vec{\epsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k},\lambda'} = \delta_{\lambda\lambda'}$$

(3a) Plug mode decomposition into $\hat{\vec{P}}$

$$\Rightarrow \hat{\vec{P}} = \int d^3x \left(\frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} + \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(+)}}{4\pi c} + h.c. \right)$$

Consider first term:

$$\int d^3x \, \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} = \sum_{\vec{k},\vec{k}',\lambda,\lambda'}{}' \frac{1}{4\pi c} \left(2\pi\hbar\sqrt{\omega_k \omega_{k'}}\right) \vec{\epsilon}_{\vec{k},\lambda} \times (\vec{e}_{\vec{k}'} \times \vec{\epsilon}^{\,*}_{\vec{k}',\lambda'}) \underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{\delta_{\vec{k},\vec{k}'}} \hat{a}_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k}',\lambda'}$$

$$= \sum_{\vec{k}} \sum_{\lambda,\lambda'}{}' \frac{\hbar\omega}{2c} \underbrace{\left[\vec{\epsilon}_{\vec{k},\lambda} \times (\vec{e}_{\vec{k}} \times \vec{\epsilon}^{\,*}_{\vec{k},\lambda'})\right]}_{\vec{e}_{\vec{k}} \underbrace{(\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda'})}_{=\delta_{\lambda\lambda'}} - \vec{\epsilon}^{\,*}_{\vec{k},\lambda'} \underbrace{(\vec{e}_{\vec{k}} \cdot \vec{\epsilon}_{\vec{k},\lambda})}_{=0}} \hat{a}_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k},\lambda'}$$

$$= \sum_{\vec{k}} \frac{\hbar \vec{k}}{2} \hat{a}_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k},\lambda}$$

having used $\vec{k} = \frac{\omega}{c} \vec{e}_{k}$

by similar steps, using $\int \frac{d^3x}{V} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} = \delta_{\vec{k}',-\vec{k}}$

$$\int d^3x \, \frac{\vec{E}^{(+)} \times \vec{B}^{(+)}}{4\pi c} = \sum_{\vec{k}}' \sum_{\lambda \lambda'}' \frac{\hbar \omega}{2c} \underbrace{\left[\vec{\epsilon}_{\vec{k},\lambda} \times (\vec{e}_{-\vec{k}} \times \vec{\epsilon}_{-\vec{k},\lambda'}) \right]}_{\vec{e}_{-\vec{k}} (\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{-\vec{k},\lambda'})} \hat{a}_{\vec{k},\lambda} \hat{a}_{-\vec{k},\lambda'}$$

Aside: $\vec{e}_{-\vec{k}} = -\vec{e}_{\vec{k}}$, thus by symmetry, when we sum over all $\vec{k}$,

$$\sum_{\lambda\lambda'} \vec{e}_{-\vec{k}} (\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{-\vec{k},\lambda'}) \xRightarrow[\vec{k} \to -\vec{k}]{} \sum_{\lambda\lambda'} \vec{e}_{\vec{k}} (\vec{\epsilon}_{-\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k},\lambda'})$$

$$= -\sum_{\lambda,\lambda'} \vec{e}_{-\vec{k}} (\vec{\epsilon}_{-\vec{k},\lambda'} \cdot \vec{\epsilon}_{\vec{k},\lambda})$$

Thus the terms cancel pairwise.

$$\Rightarrow \hat{\vec{P}} = \sum_{\vec{k},\lambda} \frac{\hbar \vec{k}}{2} \left(\hat{a}_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k},\lambda} + h.c. \right)$$

$$= \sum_{\vec{k},\lambda} \hbar \vec{k} \left(\hat{a}^{\dagger}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} + \frac{1}{2} \right)$$

But $\sum_{\vec{k}} \frac{\hbar \vec{k}}{2} = 0$ (vector's cancel)

$$\Rightarrow \boxed{\hat{\vec{P}} = \sum_{\vec{k},\lambda} \hbar \vec{k} \left(\hat{a}^{\dagger}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} \right)}$$

Neat! Momentum $= \hbar \vec{k} \times$ number of photons

(1b) Total angular momentum in field:

$$\hat{\vec{J}} = \int d^3x \;\; \vec{x} \times \hat{\vec{P}}(\vec{x})$$

where $\hat{\vec{P}}(\vec{x}) = \frac{1}{4\pi c}(\hat{\vec{E}} \times \hat{\vec{B}})$ = momentum density

Lets massage these equations a bit.

$$(\hat{\vec{E}} \times \hat{\vec{B}})_i = \epsilon_{ijk} \, E_j \, B_k \quad \text{(summation convention)}$$

$$= \epsilon_{ijk} \, E_j \, \epsilon_{klm} \, \partial_l A_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \, E_j \partial_l A_m$$

$$= E_l \, \partial_i A_l - E_l \partial_l A_i$$

Now $\vec{J} = \int d^3x \;\; \vec{x} \times \vec{P}(\vec{x})$

$$\Rightarrow \; J_j = \epsilon_{jki} \int d^3x \;\; x_k \, P_i$$

$$= \epsilon_{jki} \frac{1}{4\pi c} \int d^3x \left[E_l (x_k \partial_i) A_l - (x_k E_l)(\partial_l A_i) \right]$$

$$= \frac{1}{4\pi c} \int d^3x \; E_l \, (\vec{x} \times \vec{\nabla})_j \, A_l$$

$$+ \frac{1}{4\pi c} \int d^3x \, \epsilon_{jki} \underbrace{\partial_l (x_k E_l)}_{\left[\delta_{lk} + \vec{\nabla}\cdot\vec{E}\right]} A_i \quad \text{(integration by parts)}$$

$\vec{\nabla}\cdot\vec{E} \to 0$ in free space

$$\Rightarrow J_j = \frac{1}{4\pi c}\left(\int d^3x\ E_\ell\,(\vec{x}\times\vec{\nabla})_j\,A_\ell + \int d^3x\ (\vec{E}\times\vec{A})_j\right)$$

$$\Rightarrow \vec{J} = \vec{J}_{orb} + \vec{J}_{spin}$$

$$\boxed{\begin{aligned}\vec{J}_{orb} &= \frac{1}{4\pi c}\int d^3x\ E_\ell\,(\vec{x}\times\vec{\nabla})\,A_\ell\\ \vec{J}_{spin} &= \frac{1}{4\pi c}\int d^3x\ (\vec{E}\times\vec{A})\end{aligned}}$$

(1c)

Let us expand these terms in the plane wave basis:

$$\vec{J}_{orb} = \left(\frac{1}{4\pi c}\int d^3x\ E_\ell^{(-)}(\vec{x}\times\vec{\nabla})A_\ell^{(+)} + h.c.\right)$$
$$+\left(\frac{1}{4\pi c}\int d^3x\ E_\ell^{(+)}(\vec{x}\times\vec{\nabla})\vec{A}_\ell^{(+)} + h.c.\right)$$

Consider

$$\frac{1}{4\pi c}\int d^3x\ E_\ell^{(-)}(\vec{x}\times\vec{\nabla})A_\ell^{(+)}$$

$$= \sum_{\vec{k}\vec{k}',\lambda\lambda'} \frac{\hbar}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\ \hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k}',\lambda'}\ \vec{\epsilon}^{\,*}_{\vec{k},\lambda}\cdot\vec{\epsilon}_{\vec{k}',\lambda'} \quad \begin{pmatrix}\text{Summing}\\ \text{over } \ell\end{pmatrix}$$

$$\underbrace{\int\frac{d^3x}{V}\, e^{-i\vec{k}\cdot\vec{x}}\,(\vec{x}\times -i\vec{\nabla})\,e^{i\vec{k}'\cdot\vec{x}}}_{\mathcal{L}}$$

Aside: $\displaystyle\int \frac{d^3x}{V}\, e^{-i\vec{k}\cdot\vec{x}} \left(\vec{x} \times -i\vec{\nabla}\right) e^{i\vec{k}'\cdot\vec{x}}$

$$= \int \frac{d^3x}{V}\, e^{-i\vec{k}\cdot\vec{x}} \left(\vec{x} \times \vec{k}'\right) e^{i\vec{k}'\cdot\vec{x}}$$

$$= \left[\int \frac{d^3x}{V}\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}\; \vec{x}\right] \times \vec{k}'$$

$$= \left[i\vec{\nabla}_{\vec{k}'} \underbrace{\int d^3x\, \frac{e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}}{V}}_{\delta_{\vec{k}',\vec{k}}} \right] \times \vec{k}'$$

$$= \underbrace{\left(i\vec{\nabla}_{\vec{k}'} \times \vec{k}' \right) \delta_{\vec{k}',\vec{k}}}_{\text{derivative of delta function}} = \mathcal{d}$$

$$\frac{1}{4\pi c}\int d^3x\, E_\ell^{(-)} \left(\vec{x} \times \vec{\nabla}\right) A_\ell^{(+)}$$

$$= \sum_{\vec{k}\vec{k}',\lambda\lambda'} \frac{\hbar}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\; \hat{a}^\dagger_{\vec{k},\lambda}\, \hat{a}_{\vec{k}',\lambda'}\; \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k}',\lambda'}\; \mathcal{d}$$

$$= \sum_{\vec{k}} \frac{\hbar}{2}\, \hat{a}^\dagger_{\vec{k},\lambda} \left(i\vec{\nabla}_{\vec{k}} \times \vec{k} \right) \hat{a}_{\vec{k},\lambda} \quad \underbrace{\sum_{\lambda,\lambda'} \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k},\lambda}}_{=1}$$

Having done "integration by parts"

Consider "conjugate" term:

$$\frac{1}{4\pi c}\int d^3x \, E_\ell^{(+)} (\vec{x}\times\vec{\nabla}) A_\ell^{(+)}$$

$$= \sum_{\vec{k}\vec{k}',\lambda,\lambda'} \frac{\hbar}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}} \underbrace{\hat{a}_{\vec{k},\lambda}\hat{a}^{\dagger}_{\vec{k}',\lambda'}}_{\left(\hat{a}^{\dagger}_{\vec{k},\lambda}\hat{a}_{\vec{k}\lambda}+1\right)} \left(\vec{\epsilon}_{\vec{k}\lambda}\cdot\vec{\epsilon}^{\,*}_{\vec{k}',\lambda'}\right) \mathcal{J}^*$$

Aside $\mathcal{J}^* = +i(\nabla_{\vec{k}'}\times\vec{k}')\,\delta_{\vec{k},\vec{k}'} = -i(\nabla_{\vec{k}}\times\vec{k})\,\delta_{\vec{k},\vec{k}'}$

odd function $\vec{k}-\vec{k}'$

$$= \sum_{\vec{k},\lambda}\frac{\hbar}{2}\hat{a}^{\dagger}_{\vec{k},\lambda}\left(i\nabla_{\vec{k}}\times\vec{k}\right)\hat{a}_{\vec{k},\lambda}$$

thus

$$\int d^3x\left(E_\ell^{(-)}(\vec{x}\times\vec{\nabla})A_\ell^{(+)} + E_\ell^{(+)}(\vec{x}\times\vec{\nabla})A_\ell^{(-)}\right)$$

$$= \sum_{\vec{k},\lambda}\hat{a}^{\dagger}_{\vec{k},\lambda}\left(i\vec{\nabla}_{\vec{k}}\times\hbar\vec{k}\right)\hat{a}_{\vec{k},\lambda}$$

Co-rotating terms vanish (we'll show this explicitly for spin term)

$$\int d^3x \; E_\ell^{(+)} (\vec{x} \times \vec{\nabla}) A_\ell^{(+)} = \int d^3x \; E_\ell^{(-)} (\vec{x} \times \vec{\nabla}) A_\ell^{(-)} = 0$$

$$\Rightarrow \boxed{\hat{\vec{J}}_{\text{orbital}} = \sum_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda} \left(-i\vec{\nabla}_k \times \hbar\vec{k}\right) \hat{a}_{\vec{k},\lambda}}$$

This is the "second quantized" form of the orbital angular momentum operator in single-body quantum theory

$$\hat{\vec{L}}_{orb} = \hat{\vec{x}} \times \hat{\vec{p}} = \sum_{\vec{k},\vec{k}'} |\vec{k}\rangle\langle\vec{k}| \hat{\vec{L}}_{orb} |\vec{k}'\rangle\langle\vec{k}'|$$

(expanded in plane-wave basis)

$$= \sum_{\vec{k}} |\vec{k}\rangle \left(-i\vec{\nabla}_k \times \hbar\vec{k}\right) \langle\vec{k}|$$

where $\left.\begin{array}{l} \hat{\vec{x}} = -i\vec{\nabla}_k \\ \vec{p} = \hbar\vec{k} \end{array}\right\}$ in momentum space

Thus if we have a electromagnetic wave packet (pulse / beam) we generally carry <u>both</u> orbital and spin angular momentum.

Let's turn to the spin term...

Consider

$$\int d^3x \, \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{-i\hbar}{2} \sum_{\vec{k}\vec{k}',\lambda\lambda'} \hat{a}^{\dagger}_{\vec{k}\lambda} \hat{a}_{\vec{k}'\lambda'} \; \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \times \vec{\epsilon}_{\vec{k}',\lambda'} \underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{\to\ \delta^{(3)}(\vec{k}-\vec{k}')}$$

$$= \frac{-i\hbar}{2} \sum_{\vec{k}} \left[(\vec{\epsilon}^{\,*}_{\vec{k},+} \times \vec{\epsilon}_{\vec{k},+}) \hat{a}^{\dagger}_{\vec{k}+} \hat{a}_{\vec{k}+} + (\vec{\epsilon}^{\,*}_{\vec{k},-} \times \vec{\epsilon}_{\vec{k},-}) \hat{a}^{\dagger}_{\vec{k}-} \hat{a}_{\vec{k}-} \right]$$

Aside $\vec{\epsilon}_{\vec{k},\pm} = \frac{\vec{e}_1 \pm i\vec{e}_2}{\sqrt{2}}$ where $\vec{e}_1$ and $\vec{e}_2$ are two orthonormal vectors with $\vec{e}_1 \times \vec{e}_2 = \hat{k}$

$$\Rightarrow \vec{\epsilon}^{\,*}_{\vec{k},\pm} \times \vec{\epsilon}_{\vec{k},\pm} = \pm i\, \vec{e}_k$$

$$\Rightarrow \int d^3x \, \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}^{\dagger}_{\vec{k},+} \hat{a}_{\vec{k},+} - \hat{a}^{\dagger}_{\vec{k},-} \hat{a}_{\vec{k},-}) \vec{e}_{\vec{k}}$$

Now $$\int d^3x \, \frac{\vec{E}^{(+)} \times \vec{A}^{(-)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}_{\vec{k}+} \hat{a}^{\dagger}_{\vec{k},+} - \hat{a}_{\vec{k}-} \hat{a}^{\dagger}_{\vec{k}-}) \vec{e}_k$$

$$= \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}^{\dagger}_{\vec{k}+} \hat{a}_{\vec{k}+} - \hat{a}^{\dagger}_{\vec{k}-} \hat{a}_{\vec{k}-}) \vec{e}_k$$ (commutator cancels)

Finally note: $\vec{\epsilon}_{\vec{k},\pm} \times \vec{\epsilon}_{\vec{k},\pm} = 0$

$$\Rightarrow \int d^3x \, \vec{E}^{(+)} \times \vec{A}^{(+)} = \int d^3x \, \vec{E}^{(-)} \times \vec{A}^{(-)}$$

$$= 0$$

Thus $$\boxed{\vec{J}_{spin} = \hbar \sum_{\vec{k}} \left(\hat{a}^\dagger_{\vec{k},+} \hat{a}_{\vec{k},+} - \hat{a}^\dagger_{\vec{k},-} \hat{a}_{\vec{k},-} \right) \vec{e}_k}$$

Each photon has intrinsic "spin" angular momentum. In the circularly polarized, plane wave basis, the photon has a definite helicity, ~~carry~~ carry one hbar of angular momentum along (opposite to) the direction of propagation $\vec{e}_k$ for positive (negative) handed polarization.

The photon is spin $S=1$, yet there are only two states with definite projection of angular momentum, whereas, we might expect three $(2S+1 = 3)$. This is a very subtle point coming from the fact the the photon is massless. For more details see,

"Photons and Atoms",

(1d) Mapping photon spin onto a two-state Hilbert space

Define $\hat{\vec{J}}_{spin} = \hat{J}_x \vec{e}_x + \hat{J}_y \vec{e}_y + \hat{J}_z \vec{e}_z$

where $\hat{J}_x = \frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+)$

$\hat{J}_y = \frac{\hbar}{2i}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+)$

$\hat{J}_z = \frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)$

This is the Schwinger representation of angular momentum connecting the "Boson algebra" $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ to the angular momentum algebra $[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k$

Check: $[\hat{J}_x, \hat{J}_y] = \frac{\hbar^2}{4i}\left([\hat{a}_+^\dagger \hat{a}_-, -\hat{a}_-^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_-] \right)$

$= \frac{\hbar^2}{4i}\left[2\hat{a}_+^\dagger \hat{a}_+ (\underbrace{[\hat{a}_-^\dagger, \hat{a}_-]}_{=-1}) - 2\hat{a}_-^\dagger \hat{a}_- (\underbrace{[\hat{a}_+^\dagger, \hat{a}_+]}_{=-1})\right]$

$= i\hbar\left(\frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)\right) = i\hbar \hat{J}_z$ ✓

$[\hat{J}_x, \hat{J}_z] = \frac{\hbar^2}{4}\left([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-] \right.$

$\left. + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-] \right)$

$= \frac{\hbar^2}{4}\left(\hat{a}_+^\dagger \hat{a}_- (-1) - \hat{a}_+^\dagger \hat{a}_- (1) + \hat{a}_-^\dagger \hat{a}_+ (1) - \hat{a}_-^\dagger \hat{a}_+ (-1)\right)$

$= \frac{-\hbar^2}{2}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+) = -i\hbar \hat{J}_y$ ✓

$$[\hat{J}_y, \hat{J}_z] = \frac{\hbar^2}{4i}\Big([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-]$$
$$- [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-]\Big)$$

$$= \frac{\hbar^2}{4i}\left(\hat{a}_+^\dagger \hat{a}_-(-1) - \hat{a}_+^\dagger \hat{a}_-(1) - \hat{a}_-^\dagger \hat{a}_+(1) + \hat{a}_-^\dagger \hat{a}_+(-1)\right)$$

$$= -\frac{\hbar^2}{2i}\left(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+\right)$$

$$= i\hbar\left[\frac{\hbar}{2}\left(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+\right)\right] = i\hbar \hat{J}_x \quad \checkmark$$

The Schwinger representation is the "second quantized form" of the spin ½ operators

$$\hat{J}_x = \frac{\hbar}{2}\left(|+_z\rangle\langle -_z| + |-_z\rangle\langle +_z|\right)$$
$$\hat{J}_y = \frac{\hbar}{2i}\left(|+_z\rangle\langle -_z| - |-_z\rangle\langle +_z|\right)$$
$$\hat{J}_z = \frac{\hbar}{2}\left(|+_z\rangle\langle +_z| - |-_z\rangle\langle -_z|\right)$$

"Second quantize" $|\pm_z\rangle \Rightarrow \hat{a}_\pm^\dagger$ create spin up or down

$\langle \pm_z| \Rightarrow \hat{a}_\pm$ annihilate spin up or down

thus, we can easily map the spin angular momentum of the ~~th~~ photon onto the Bloch sphere, also known as the Poincaré sphere as we visited in PS# 1