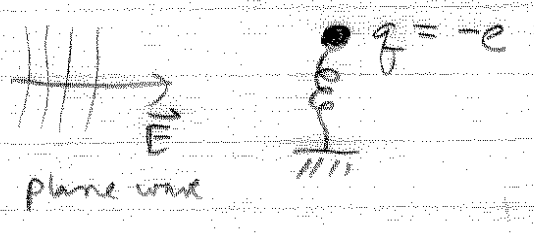
Physics 566 - Quantum Optics

Problem Set #5 Solutions

Problem 1: Lorentz Classical Model



charge on spring

Eq of motion $\quad m\ddot{\vec{x}} = -m\Gamma\dot{\vec{x}} - m\omega_0^2\vec{x} - e\vec{E}(t)$

Rate at which field does work

$$\frac{dW}{dt} = \dot{\vec{x}}\cdot\vec{F} = -e\dot{\vec{x}}\cdot\vec{E}$$

Need to solve for $\dot{\vec{x}}$ in steady state.

Use complex representation: $\vec{E} = \vec{E}_0 e^{-i\omega t}$

$$\Rightarrow \text{steady state} \quad \vec{x} = \vec{x}_0 e^{-i\omega t}$$

$$\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0^2\vec{x} = -\frac{e}{m}\vec{E}_0 e^{-i\omega t}$$

$$\Rightarrow \left(-\omega^2 + \omega_0^2 - i\omega\Gamma\right)\vec{x}_0 = -\frac{e}{m}\vec{E}_0$$

$$\Rightarrow \quad \vec{x}_0 = \left(\frac{-e/m}{-\omega^2 + \omega_0^2 - i\omega\Gamma}\right)\vec{E}_0$$

Go to near resonance limit $\quad \omega - \omega_0 \equiv \Delta$

$$\Rightarrow \quad \omega_0 = \omega - \Delta \qquad \omega_0^2 - \omega^2 = (\omega - \Delta)^2 - \omega^2$$

$$\approx -2\omega\Delta \quad \text{to } \mathcal{O}(\Delta)$$

$$\Rightarrow \quad \vec{x}_0 \approx \left(\frac{e/2m\omega}{\Delta + i\frac{\Gamma}{2}} \right) \vec{E}_0$$

$\therefore$ Time averaged over a period

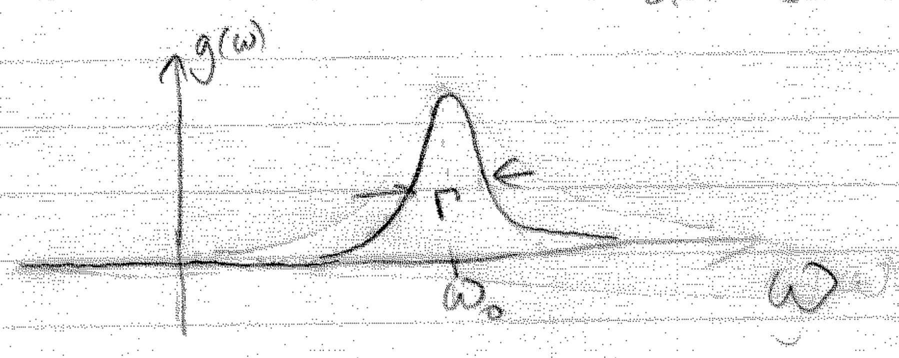
$$\frac{dW}{dt} = -\frac{e}{2}\,\mathrm{Re}\left(-i\omega\, \vec{x}_0 \cdot \vec{E}_0\right)$$

$$= -\frac{e\omega}{2}\,\mathrm{Im}\left(\vec{x}_0 \cdot \vec{E}_0\right)$$

$$= -\frac{e\omega}{2}\left(\frac{e}{2m\omega}\right)\frac{-\frac{\Gamma}{2}}{\Delta^2 + \frac{\Gamma^2}{4}}\,|\vec{E}_0|$$

$$= \frac{\pi e^2}{4m}\underbrace{\left(\frac{\frac{\Gamma}{2\pi}}{\Delta^2 + \frac{\Gamma^2}{4}}\right)}_{g(\omega)}|\vec{E}_0|^2$$

$g(\omega) \leftarrow$ atomic line-shape



(b) Absorption cross-section

$$P_{abs} = \sigma_{abs} I_{inc} = \sigma_{abs} \frac{c}{8\pi} |\vec{E}_0|^2$$

$$\parallel$$

$$\frac{dW_{abs}}{dt} = \frac{\pi e^2}{4m} g(\omega) |\vec{E}_0|$$

$$\Rightarrow \boxed{\sigma_{abs} = \frac{2\pi^2 e^2}{mc} g(\omega)}$$

Evaluate for $\frac{\Gamma}{2\pi} = 10\ MHz$ $\quad \lambda = 589\ nm$

on resonance $\Delta = 0$ $\quad g(\omega = \omega_0) = \frac{2}{\pi} \frac{1}{\Gamma}$

$$\Rightarrow \sigma_{abs} = 2\left(\frac{2\pi}{\Gamma}\right) \frac{e^2}{mc} = 2\left(\frac{2\pi}{\Gamma}\right) c \underbrace{\left(\frac{e^2}{mc^2}\right)}_{\text{classical electron radius} = r_e}$$

$$= 2\left(10^{-7} s^{+1}\right)\left(3 \times 10^{10} \frac{cm}{s}\right)\left(2.8 \times 10^{-13}\ cm\right)$$

$$\boxed{\sigma_{abs} = 1.68 \times 10^{-9}\ cm^2}$$

Note: On resonance $\sigma_{abs} \sim \lambda_0^2 = 3.5 \times 10^{-9}\ cm^2$

(c) $\sigma_{quantum} = 4\pi^2 \frac{e^2}{\hbar c} |\langle e|\vec{x}|g\rangle|^2 \omega\, g(\omega)$

Oscillator strength $f = \left.\frac{\sigma_{quantum}}{\sigma_{classical}}\right|_{resonance}$

$$\Rightarrow \boxed{f = \frac{2m\omega_0}{\hbar} |\langle e|\vec{x}|g\rangle|^2}$$

(d) The driven oscillator will radiate electromagnetically

$$P_{radiated} = \frac{ck^4}{3} |\vec{d}_0|^2 \quad \text{Larmor formula}$$

$$= \frac{ck^4 e^2}{3} |\vec{x}_0|^2$$

$$P_{abs} = \frac{e\omega}{2} \operatorname{Im}(\vec{x}_0 \cdot \vec{E}_0)$$

Now $\vec{x}_0 = \alpha \vec{E}_0$ where $\alpha = \frac{e/2m\omega}{\Delta + i\frac{\Gamma}{2}}$

$$\Rightarrow \frac{ck^4 e^2}{3} |\vec{E}_0|^2 \underbrace{|\alpha|^2}_{\downarrow\ \frac{\frac{e^2}{4m^2\omega^2}}{\Delta^2 + \frac{\Gamma^2}{4}}} = \frac{e\omega}{2} |\vec{E}_0|^2 \underbrace{\operatorname{Im}(\alpha)}_{\downarrow\ \frac{-i\frac{\Gamma}{2}\, e/2m\omega}{\Delta^2 + \frac{\Gamma^2}{4}}}$$

$$\therefore \boxed{\Gamma_{class} = \frac{2}{3}\frac{ck^4}{\omega^2}\frac{e^2}{m} = \frac{2}{3}\frac{e^2}{mc^3}\omega^2}$$

(e) We have quantum mechanically,

$$\Gamma_{quantum} = \frac{4}{3}\frac{k^3}{\hbar}|\langle e|\vec{d}_0|g\rangle|^2$$

$$= \frac{4}{3}\frac{e^2}{\hbar}|\langle e|\vec{x}_0|g\rangle|^2 \frac{\omega^3}{c^3}$$

$$\rightarrow \frac{\Gamma_{quant}}{\Gamma_{class}} = \frac{2m\omega}{\hbar}|\langle e|\vec{x}_0|g\rangle|^2 = f \checkmark$$

the oscillator strength

and

$$\sigma_{\text{classical}} = \frac{2\pi^2 e^2}{mc}\, g(\omega) \quad ; \quad \sigma_{\text{quantum}} = \frac{4\pi^2 e^2 \omega_L}{\hbar c} |\langle e|\vec{x}|g\rangle|^2\, g(\omega_L)$$

when
$$g(\omega) = \frac{\Gamma/2\pi}{(\omega_L-\omega_0)^2 + \Gamma^2/4}$$

write

$$\sigma_{\text{classical}} = \frac{2\pi^2 e^2}{mc}\, \frac{\Gamma_{\text{classical}}/2\pi}{(\omega_L-\omega_0)^2+\Gamma^2/4} \quad ; \quad \sigma_{\text{quantum}} = \frac{2\pi^2 e^2}{mc}\, \frac{\Gamma_{\text{quantum}}/2\pi}{(\omega_L-\omega_0)^2+\Gamma^2/4}$$

$$\Rightarrow \frac{\Gamma_{\text{quantum}}}{\Gamma_{\text{classical}}} = f$$

3. a. $$\hat{H} = \frac{1}{2}\hbar\omega_{eg}\hat{\sigma}_z + \sum_{\vec{k},h}\hbar\omega_k \hat{a}^\dagger_{\vec{k}h}\hat{a}_{\vec{k}h} - \sum_{\vec{k},h}\hbar\left(g_{\vec{k}h}\hat{\sigma}_+\hat{a}_{\vec{k}h} + g^*_{\vec{k}h}\hat{a}^\dagger_{\vec{k}h}\hat{\sigma}_-\right)$$

where $$g_{\vec{k}h} = i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\; \vec{\epsilon}_{\vec{k}h}\cdot\vec{d}_{eg}$$

$$\dot{\hat{a}}_{\vec{k},h} = -i/\hbar \left[\hat{a}_{\vec{k}h}, \hat{H}\right]$$

henceforth
drop the hats on operators

$$= -i\sum_{\vec{k}',h'}\omega_{k'}\left[a_{\vec{k}h}, a^\dagger_{\vec{k}'h'}a_{\vec{k}'h'}\right] + i\sum_{\vec{k}',h'} g^*_{\vec{k}',h'}\left[a_{\vec{k}h}, a^\dagger_{\vec{k}',h'}\right]\sigma_-$$

$$-i\,\underbrace{\frac{1}{2}\omega_{eg}\left[a_{\vec{k},h},\sigma_z\right]}_{0} + i\sum_{\vec{k}',h'} g_{\vec{k}'h'}\,\sigma_+\,\cancel{\left[a_{\vec{k}h}, a_{\vec{k}'h'}\right]}^{\,0}$$

Field and atomic
operators commute
at equal times

$$\dot{a}_{\vec{k}h} = -i \sum_{\vec{k}',h'} \omega_{k'} \Big(\underbrace{[a_{\vec{k}h}, a^{+}_{\vec{k}'h'}]}_{=\delta_{kk'}\delta_{hh'}} a_{\vec{k}'h'} + a^{+}_{\vec{k}'h'} [a_{\vec{k}h}, a_{\vec{k}'h'}] \Big) + i \sum_{\vec{k}',h'} g^{*}_{\vec{k}'h'} \delta_{kk'} \delta_{hh'}$$

$$\boxed{\dot{a}_{\vec{k},h} = -i\omega_k a_{\vec{k}h} + i g^{*}_{\vec{k}h} \sigma_-}$$

$$\dot{\sigma}_- = -i/\hbar \, [\sigma_-, H]$$

Again using equal time commutation

$$= -i \frac{\omega_{eg}}{2} \underbrace{[\sigma_-, \sigma_z]}_{=2\sigma_-} + i \sum_{\vec{k},h} \Big(g_{\vec{k},h} \underbrace{[\sigma_-, \sigma_+]}_{=-\sigma_z} a_{\vec{k}h} + g^{*}_{\vec{k},h} a^{+}_{\vec{k},h} \cancel{[\sigma_-, \sigma_-]}^{0} \Big)$$

$$\boxed{\dot{\sigma}_- = -i\omega_{eg} \sigma_- - i \sum_{\vec{k},h} g_{\vec{k}h} \sigma_z a_{\vec{k}h}}$$

$$\dot{\sigma}_z = -i/\hbar \, [\sigma_z, H]$$

$$= -i \frac{\omega_{eg}}{2} \cancel{[\sigma_z, \sigma_z]}^{0} + i \sum_{\vec{k},h} \Big(g_{\vec{k},h} \underbrace{[\sigma_z, \sigma_+]}_{=2\sigma_+} a_{\vec{k},h} + g^{*}_{\vec{k},h} a^{+}_{\vec{k},h} \underbrace{[\sigma_z, \sigma_-]}_{=-2\sigma_-} \Big)$$

$$\boxed{\dot{\sigma}_z = 2i \sum_{\vec{k},h} \left(g_{\vec{k},h} \sigma_+ a_{\vec{k},h} - g^{*}_{\vec{k},h} a_{\vec{k},h} \sigma_- \right)}$$

b. Solve

$$\dot{a}_{\vec{k},h} = -i\omega_k a_{\vec{k},h} + i g^{*}_{\vec{k},h} \sigma_-$$

$$\dot{a}_{\vec{k},h} + i\omega_k a_{\vec{k},h} = i g^{*}_{\vec{k},h} \sigma_-$$

Multiply both sides by $e^{i\omega_k t}$

$$e^{i\omega_k t}\left(\dot{a}_{\vec{k},h} + i\omega_k a_{\vec{k},h}\right) = i g^*_{\vec{k},h}\,\sigma_-\, e^{i\omega_k t}$$

$$\frac{d}{dt}\left(a_{\vec{k},h}\, e^{i\omega_k t}\right) = e^{i\omega_k t}\, i g^*_{\vec{k},h}\,\sigma_-$$

$$a e^{i\omega_k t} = i g^*_{\vec{k},h}\int_0^t e^{i\omega_k t'}\,\sigma_-(t')\,dt' + C \longleftarrow \text{Integration constant}$$

$$a(t) = i g^*_{\vec{k},h}\int_0^t e^{-i\omega_k(t-t')}\,\sigma_-(t')\,dt' + C e^{-i\omega_k t}$$

At $t=0$ $\quad a_{\vec{k},h}(t) \to a_{\vec{k},h}(0)$

$$\Rightarrow \boxed{a_{\vec{k},h}(t) = \underbrace{a_{\vec{k},h}(0)\, e^{-i\omega_k t}}_{a^{free}_{\vec{k},h}(t)} + \underbrace{i g^*_{\vec{k},h}\int_0^t dt'\,\sigma_-(t')\, e^{-i\omega_k(t-t')}}_{a^{source}_{\vec{k},h}(t)}}$$

The source component is due to dipole radiation by the atom

We can check that this is the soln

$$\dot{a}_{\vec{k},h}(t) = -i\omega_k\, a_{\vec{k},h}(0)\, e^{-i\omega_k t} + i g^*_{\vec{k},h}\,\frac{d}{dt}\left[\left(\int_0^t dt'\,\sigma_-(t')\, e^{i\omega_k t'}\right) e^{-i\omega_k t}\right]$$

$$\parallel$$

$$\left(-i\omega_k\int_0^t dt'\,\sigma_-(t')\, e^{i\omega_k t'}\right) e^{-i\omega_k t} + e^{-i\omega_k t}\,\sigma_-(t)\, e^{i\omega_k t}$$

$$\dot{a}_{\vec{k},h}(t) = -i\omega_k\left(a_{\vec{k},h}(0)\, e^{-i\omega_k t} + i g^*_{\vec{k},h}\int_0^t dt'\,\sigma_-(t')\, e^{-i\omega_k(t-t')}\right) + i g^*_{\vec{k},h}\,\sigma_-(t)$$

$$\dot{a}_{\vec{k},h}(t) = -i\omega_k\, a_{\vec{k},h}(t) + i g^*_{\vec{k},h}\,\sigma_-(t) \qquad \checkmark$$

c. $$\left[a_{\vec{k}h}(t), a^{+}_{\vec{k}'h'}(t)\right] = \left[U^{+}(t)\, a_{\vec{k}h}(0)\, U(t),\ U^{+}(t)\, a^{+}_{\vec{k}'h'}(0)\, U(t)\right]$$

where $U(t) = e^{-iHt/\hbar}$

$$= U^{+}(t)\left[a_{\vec{k}h}(0), a^{+}_{\vec{k}'h'}(0)\right] U(t)$$

$$= U^{+}(t)\, \delta_{kk'}\delta_{hh'}\, U(t)$$

$$\left[a_{\vec{k}h}(t), a^{+}_{\vec{k}'h'}(t)\right] = \delta_{kk'}\delta_{hh'}$$

The source part alone does not satisfy this relation

$$\left[a^{source}_{\vec{k}h}(t), a^{+source}_{\vec{k}'h'}(t)\right] = g^{*}_{\vec{k}h} g_{\vec{k}'h'}\left[\int_0^t dt'\, \sigma_-(t')\, e^{-i\omega_k(t-t')}, \int_0^t dt''\, \sigma_+(t'')\, e^{+i\omega_{k'}(t$$

$$\neq \delta_{kk'}\delta_{hh'}$$

d. $$\dot{\sigma}_z(t) = 2i\left[\sum_{\vec{k},h} g_{\vec{k}h}\, \sigma_+(t)\, a_{\vec{k}h}(0)\, e^{-i\omega_k t} + i|g_{\vec{k},h}|^2\int_0^t dt'\, \sigma_+(t)\, \sigma_-(t')\, e^{-i\omega_k(t-t')}\right.$$
$$\left. - g^{*}_{\vec{k},h}\, a^{+}_{\vec{k},h}(0)\, e^{i\omega_k t}\, \sigma_-(t) + i|g_{\vec{k},h}|^2\int_0^t dt'\, \sigma_+(t')\, \sigma_-(t)\, e^{i\omega_k(t-t')}\right]$$

For the Heisenberg state $|\Psi\rangle = |\psi\rangle_{atom} \otimes |0\rangle_{field}$ the terms

$$\langle\Psi|\, \sigma_+(t)\, a_{\vec{k}h}(0)\,|\Psi\rangle = \langle\Psi|\, a^{+}_{\vec{k}h}(0)\, \sigma_-(t)\,|\Psi\rangle = 0$$

because the field is a vacuum state.

$$\Rightarrow \langle\Psi|\,\dot{\sigma}_z(t)\,|\Psi\rangle = \frac{d}{dt}\langle\sigma_z\rangle = -2\sum_{\vec{k},h}|g_{\vec{k},h}|^2\int_0^t dt'\Big(\langle\sigma_+(t)\,\sigma_-(t')\rangle e^{-i\omega_k(t-t')}$$
$$+ \langle\sigma_+(t')\,\sigma_-(t)\rangle e^{i\omega_k(t-t')}\Big)$$

$$\frac{d}{dt}\langle\sigma_z\rangle = -2\sum_{\vec{k},h}|g_{\vec{k},h}|^2\int_0^t\left(\langle\sigma_+(t)\sigma_-(t')\rangle e^{-i\omega_k(t-t')} + c.c.\right)dt'$$

$$\boxed{\frac{d}{dt}\langle\sigma_z\rangle = -2\sum_{\vec{k},h}|g_{\vec{k},h}|^2\int_0^t dt'\langle\sigma_+(t)\sigma_-(t')\rangle e^{-i\omega_k(t-t')} + c.c.}$$

using $\langle\sigma_+(t)\sigma_-(t')\rangle^* = \langle(\sigma_+(t)\sigma_-(t'))^\dagger\rangle$
$= \langle\sigma_-(t')^\dagger\sigma_+(t)^\dagger\rangle = \langle\sigma_+(t')\sigma_-(t)\rangle$

a. Under the Markov approximation,

$\sigma_-(t) = \Sigma_-(t)e^{-i\omega_{eg}t}$ where $\Sigma_-(t)$ is slowly varying in the scale of ω_{eg}

then $\langle\Sigma_+(t)\Sigma_-(t')\rangle \simeq \langle\Sigma_+(t)\Sigma_-(t)\rangle = \langle\sigma_+(t)\sigma_-(t)\rangle$

$$\frac{d}{dt}\langle\sigma_z\rangle = -2\sum_{\vec{k},h}|g_{\vec{k},h}|^2\int_0^t dt'\,\langle\Sigma_+(t)\Sigma_-(t')\rangle e^{i\omega_{eg}(t-t')}e^{-i\omega_k(t-t')} + c.c.$$

$$\simeq -2\sum_{\vec{k},h}|g_{\vec{k},h}|^2\int_0^t dt'\,\langle\sigma_+(t)\sigma_-(t)\rangle e^{i(\omega_{eg}-\omega_k)(t-t')} + c.c.$$

Now
$$\int_0^T e^{\pm i(\omega_{eg}-\omega_k)(t-t')}dt' = \pi\delta(\omega_{eg}-\omega_k) \pm i\underbrace{P\left(\frac{1}{\omega_{eg}-\omega_k}\right)}$$

Cauchy principal part
responsible for level shifts

$$\frac{d}{dt}\langle\sigma_z\rangle = -2\sum_{\vec{k},h}|g_{\vec{k},h}|^2\langle\sigma_+(t)\sigma_-(t)\rangle\left(\pi\delta(\omega_{eg}-\omega_k) + iP\left(\frac{1}{\omega_{eg}-\omega_k}\right) + \pi\delta(\omega_{eg}-\omega_k) - iP\left(\frac{1}{\omega_{eg}-\omega_k}\right.\right.$$

$$\frac{d}{dt}\langle \sigma_z \rangle = -2 \underbrace{\sum_{\vec{k},\lambda} 2\pi\, \delta(\omega_{eg} - \omega_k)\, |g_{\vec{k},\lambda}|^2}_{=\,\Gamma} \langle \sigma_+(t)\sigma_-(t) \rangle$$

$$= -2\Gamma \langle \sigma_+(t)\sigma_-(t) \rangle$$

$$= -2\Gamma \left\langle \frac{1 + \sigma_z(t)}{2} \right\rangle$$

$$\boxed{\frac{d}{dt}\langle \sigma_z \rangle = -\Gamma \langle 1 + \sigma_z(t) \rangle}$$

Physics 566

Problem Set #4 Solutions

Problem 1: Boson Algebra

(a) $\int \frac{d^2\beta}{\pi} e^{-A|\beta|^2} e^{\alpha\beta^* - \alpha^*\beta}$ Let $\alpha = X_1 + iX_2$, $\beta = Y_1 + iY_2$

$$\int \frac{dY_1\,dY_2}{\pi} e^{-A(Y_1^2+Y_2^2)} e^{2i(X_2Y_1 - X_1Y_2)}$$

$$= \underbrace{\left(\int_{-\infty}^{\infty} \frac{dY_1}{\sqrt{\pi}} e^{-AY_1^2} e^{2iX_2Y_1}\right)}_{\frac{e^{-X_2^2/A}}{\sqrt{A}}} \underbrace{\left(\int_{-\infty}^{\infty} \frac{dY_2}{\sqrt{\pi}} e^{-AY_2^2} e^{2iX_1Y_2}\right)}_{\frac{e^{-X_1^2/A}}{\sqrt{A}}}$$

(Gaussian Fourier Integral)

$$= \frac{1}{A} e^{-|\alpha|^2/A}$$

(b) $$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \sum_{n,m} \int \frac{d^2\alpha}{\pi} \left(\frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}\right)\left(\frac{\alpha^{*m}}{\sqrt{m!}} e^{-|\alpha|^2/2}\right) |n\rangle\langle m|$$

Change to polar coords: $\alpha = Ae^{i\theta} \Rightarrow d^2\alpha = A\,d\theta\,dA$

$$\Rightarrow \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \sum_{n,m} \frac{|n\rangle\langle m|}{\pi\sqrt{n!\,m!}} \int_0^\infty A\,dA\, e^{-A^2} A^{n+m} \underbrace{\int_0^{2\pi} d\theta\, e^{i(n-m)\theta}}_{2\pi\,\delta_{nm}}$$

$$= \sum_{n=0}^{\infty} \frac{2|n\rangle\langle n|}{n!} \underbrace{\int_0^\infty dA\, A^{2n+1} e^{-A^2}}_{n!/2 \text{ (Gamma function)}}$$

$$= \sum_{n} |n\rangle\langle n| = \mathbb{1} \checkmark$$

(c) Quadrature operators: $\hat{a} = \hat{X}_1 + i\hat{X}_2$

$$\Rightarrow \hat{X}_1 = (\hat{a} + \hat{a}^\dagger)/2 \qquad \hat{X}_2 = (\hat{a} - \hat{a}^\dagger)/2i$$

Under $\hat{U} = e^{-i\theta \hat{a}^\dagger \hat{a}}$ $\quad \hat{U}^\dagger \hat{a} \hat{U} = \hat{a} e^{-i\theta}$

$\uparrow$ Free Hamiltonian $\quad \hat{U}^\dagger \hat{a}^\dagger \hat{U} = \hat{a}^\dagger e^{+i\theta}$

$$\Rightarrow \hat{U}^\dagger \hat{X}_1 \hat{U} = \frac{1}{2}(\hat{U}^\dagger \hat{a} \hat{U} + h.c.) = \frac{1}{2}(\hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta})$$

$$= \mathrm{Re}(\hat{a} e^{-i\theta}) = \hat{X}_1 \cos\theta + \hat{X}_2 \sin\theta$$

$$\hat{U}^\dagger \hat{X}_2 \hat{U} = \mathrm{Im}(\hat{a} e^{-i\theta}) = \hat{X}_2 \cos\theta - \hat{X}_1 \sin\theta$$

The free evolution operator corresponds to a rotation in phase space

(d) Consider $\hat{D}(\alpha+\beta) = \exp\{(\alpha+\beta)\hat{a}^\dagger - (\alpha^* + \beta^*)\hat{a}\}$

Aside: Recall Baker-Campbell-Hausdorff

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

when ($[A,B]$ commutes with A and B)

$$\Rightarrow \hat{D}(\alpha+\beta) = \hat{D}(\alpha)\hat{D}(\beta) \exp\left\{-\frac{1}{2}\left[\alpha\hat{a}^\dagger - \alpha^*\hat{a}, \beta\hat{a}^\dagger - \beta^*\hat{a}\right]\right\}$$

$$\hat{D}(\alpha+\beta) = \hat{D}(\alpha)\hat{D}(\beta) \exp\{-\frac{1}{2}\underbrace{(\alpha\beta^* - \alpha^*\beta)}_{i\,\mathrm{Im}(\alpha\beta^*)}\}$$

$$\Rightarrow \boxed{\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta) \exp\{i\,\mathrm{Im}(\alpha\beta^*)\}}$$

(e) Matrix elements of displacement operator

• Vacuum: $\langle 0|\hat{D}(\alpha)|0\rangle = \langle 0| e^{\alpha \hat{a}^{\dagger} - \alpha^{*}\hat{a}} |0\rangle$

$$= \underbrace{\langle 0| e^{\alpha \hat{a}^{\dagger}} e^{-\alpha^{*}\hat{a}} |0\rangle}_{=1} \, e^{-|\alpha|^2/2} \qquad \left(\begin{matrix}\text{Normal}\\ \text{order}\end{matrix}\right)$$

• Coherent States: $\langle \alpha_1 | \hat{D}(\alpha) | \alpha_2 \rangle = \langle 0| \hat{D}^{\dagger}(\alpha_1)\, \hat{D}(\alpha)\, \hat{D}(\alpha_2) |0\rangle$

$$= \langle 0| \hat{D}(-\alpha_1)\, \hat{D}(\alpha)\, \hat{D}(\alpha_2)|0\rangle = \langle 0| \hat{D}^{\dagger}(\alpha_1)\, \hat{D}(\alpha+\alpha_2)|0\rangle e^{i\,\mathrm{Im}(\alpha\alpha_2^{*})}$$

$$= \langle 0| \hat{D}(\alpha+\alpha_2-\alpha_1)|0\rangle \exp\{ i\,\mathrm{Im}(\alpha\alpha_2^{*}) - i\,\mathrm{Im}(\alpha_1(\alpha^{*}+\alpha_2^{*}))\}$$

$$= \exp\left\{ \frac{-|\alpha+\alpha_2-\alpha_1|^2}{2} \right\} \exp\{ i\,\mathrm{Im}(\alpha\alpha_2^{*} - \alpha_1\alpha^{*} - \alpha_1\alpha_2^{*})\}$$

• Fock states: $\langle n|\hat{D}(\alpha)|n\rangle = \langle n| e^{\alpha\hat{a}^{\dagger}} e^{-\alpha^{*}\hat{a}} |n\rangle \, e^{-|\alpha|^2/2}$

$$= \sum_{m m'} (-1)^m \frac{\langle n| (\hat{a}^{\dagger})^m (\hat{a})^{m'} |n\rangle}{m!\, m'!} (\alpha^{*})^m (\alpha)^{m'} e^{-|\alpha|^2/2}$$

$$= \underbrace{\left(\sum_{m=0}^{\infty} (-1)^m \frac{n!}{(n-m!)} \frac{1}{(m!)^2} (|\alpha|^2)^m \right)}_{\mathcal{L}_n(|\alpha|^2)} e^{-|\alpha|^2/2}$$

see Arfken (13.32) 3rd edition

$$\Rightarrow \quad \langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\, \mathcal{L}_n(|\alpha|^2)$$