Physics 566, Quantum Optics
Problem Set #6 Solutions

Problem 2: Properties of the Wigner function

$$W(X_1, X_2) = \int \frac{dX'}{\pi} e^{-2iX'X_2} \langle X_1 - X' | \hat{\rho} | X_1 + X' \rangle$$

$$= \int \frac{dX'}{\pi} e^{-2iX'X_2} \psi(X_1 - X') \psi^*(X_1 + X')$$

The last form is what Wigner originally wrote down in considering the phase space description of ordinary quantum mechanics.

(a) Marginals: $P(X_1) = \int dX_2 \, W(X_1, X_2)$

$P(X_2) = \int dX_1 \, W(X_1, X_2)$

We seek to show $P(X_1) = \langle X_1 | \hat{\rho} | X_1 \rangle = \left(|\psi(X_1)|^2 \right.$ Pure

$P(X_2) = \langle X_2 | \hat{\rho} | X_2 \rangle = \left. |\tilde{\psi}(X_2)|^2 \right)$ States

$$\int dX_2 \, W(X_1, X_2) = \int \frac{dX'}{\pi} \underbrace{\int e^{-2iX'X_2} dX_2}_{2\pi \delta(2X') = \pi \delta(X')} \langle X_1 - X' | \hat{\rho} | X_1 + X' \rangle$$

$$= \langle X_1 | \hat{\rho} | X_1 \rangle \quad \checkmark$$

$$\int dX_1 \, W(X_1, X_2) = \int \frac{dX'}{\pi} e^{-2iX'X_2} \int dX_1 \langle X_1 - X' | \hat{\rho} | X_1 + X' \rangle$$

Aside $\int dX_1 \langle X_1 - X' | \hat{\rho} | X_1 + X' \rangle = \int dP\, dP' \int \langle X_1 - X' | P \rangle \underbrace{\langle P | \hat{\rho} | P' \rangle} \langle P' | X_1 + X' \rangle \, dX_1$

insert complete moment set

$$= \int dP\, dP' \langle P | \hat{\rho} | P' \rangle \int \frac{dX_1}{2\pi} e^{i(X_1 - X')P} e^{-i(X_1 + X')P'}$$

(c) suppose the operator is only a function of the general quadrature operators $\hat{X}(\theta) = \frac{a e^{-i\theta} \pm \hat{a}^{\dagger} e^{i\theta}}{(i)}$

$$\hat{f} \equiv f(\hat{X}_i(\theta)) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}\Big|_0 \left(\hat{X}_i(\theta)\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}\Big|_0 \left(\hat{a} e^{-i\theta} + \hat{a}^{\dagger} e^{i\theta}\right)^n = \text{symmetrically ordered product}$$

$$\Longrightarrow \langle \hat{f}_i \rangle = \int dX_1(\theta)\, dX_2(\theta)\; f(X_i(\theta))\; W(X_1(\theta), X_2(\theta))$$

$\Longrightarrow$ $W(X_1(\theta), X_2(\theta))$ is the appropriate quasiprobability distribution when considering measurements of quadratures

Problem 3: The "Schrödinger cat" state

Given superposition of two coherent states:

$$|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle) \qquad N = \frac{1}{\sqrt{2(1+e^{-|\alpha_1-\alpha_2|^2}}}$$

Specialize to case $\alpha_1 = -\alpha_2 = \alpha_0$ (Real)

The Wigner function

$$W(\alpha) = \int \frac{d\beta}{\pi^2}\, \chi(\beta)\, e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi(\beta) = \mathrm{Tr}(\hat\rho\, \hat D(\beta)) = \langle\psi|\hat D(\beta)|\psi\rangle$$

$$= N\Big(\langle\alpha_0|\hat D(\beta)|\alpha_0\rangle + \langle-\alpha_0|\hat D(\beta)|-\alpha_0\rangle$$

$$\langle-\alpha_0|\hat D(\beta)|\alpha_0\rangle + \langle\alpha_0|\hat D(\beta)|-\alpha_0\rangle\Big)$$

The first two terms represent the characteristic function of the coherentstates $|\alpha_0\rangle$ and $|-\alpha_0\rangle$ respectively. $W_{\pm\alpha_0}(\alpha) = \frac{2}{\pi} e^{-2|\alpha\pm\alpha_0|^2}$

To calculate the off-diagonal terms, use part 1(c)

$$\langle-\alpha_0|\hat D(\beta)|\alpha_0\rangle = e^{-|\beta+2\alpha_0|^2/2}$$

$$\int \frac{d\beta}{\pi^2}\, e^{-|\beta+2\alpha_0|^2}\, e^{\alpha\beta^*-\alpha^*\beta}$$

let $\gamma = \beta + 2\alpha_0 \implies \beta = \gamma - 2\alpha_0$

$$\underbrace{\int \frac{d\gamma}{\pi^2}\, e^{-\frac{|\gamma|^2}{2}}\, e^{\alpha\gamma^*-\alpha^*\gamma}}_{=(2/\pi)e^{-2|\alpha|^2}\text{ (part 1a)}}\, e^{-2\alpha_0(\alpha-\alpha^*)} = \frac{2}{\pi} e^{-2|\alpha|^2}\, e^{-4i\alpha_0 \mathrm{Im}(\alpha)}$$

Putting it all together,

$$\boxed{W(\alpha,\alpha^*) = \frac{2N}{\pi}\left\{ e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2} + + 2e^{-2|\alpha|^2}\cos\left(4\alpha_0\,\mathrm{Im}(\alpha)\right)\right\}}$$

The Wigner function is <u>negative</u> for some α

$\Longrightarrow$ <u>Nonclassical state</u>

Let $\alpha = \quad + i P$

$$\Rightarrow W(\ , P) = \frac{2N}{\pi}\left\{ e^{-2\{(x-\alpha_0)^2+p^2\}} + e^{-2\{(x+\alpha_0)^2+p^2\}} + 2e^{-2(x^2+p^2)}\cos(4\alpha_0 p)\right\}$$

Now $\int_{-\infty}^{\infty} dP\, W(x,P) = N\sqrt{\frac{2}{\pi}}\left(e^{-2(x-\alpha_0)^2} + e^{-2(x+\alpha_0)^2} + e^{-2(x^2+\alpha_0^2)}\right)$

$\stackrel{?}{=} |\langle x|\psi\rangle|^2 = |\psi(x)|^2$

Yes given $\langle x|\alpha_0\rangle = \left(\frac{2}{\pi}\right)^{1/4} e^{-(x-\alpha_0)^2}$ in our unit

$$\int_{-\infty}^{\infty} dx\, W(x,P) = N\sqrt{\frac{2}{\pi}}\left(2e^{-2p^2}\right)\cos(4\alpha_0 p)$$

oscillation from superposition

$$= |\tilde{\psi}(p)|^2$$

Yes ✓

Following are plots of the Wigner function for $\alpha_0 = 1, 5, 10$. We see two kinds of feature

(1) Gaussian packets localized near $\pm\alpha_0$

(2) Oscillations as a function of P near $x=0$

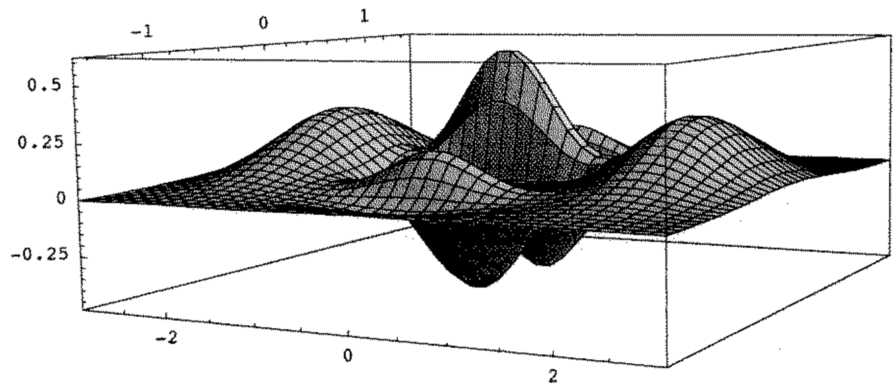
Feature (1) is expected classically in terms of a distribution in phase space

Feature (2) is the nonclassical interference coming from the "quantum superposition"

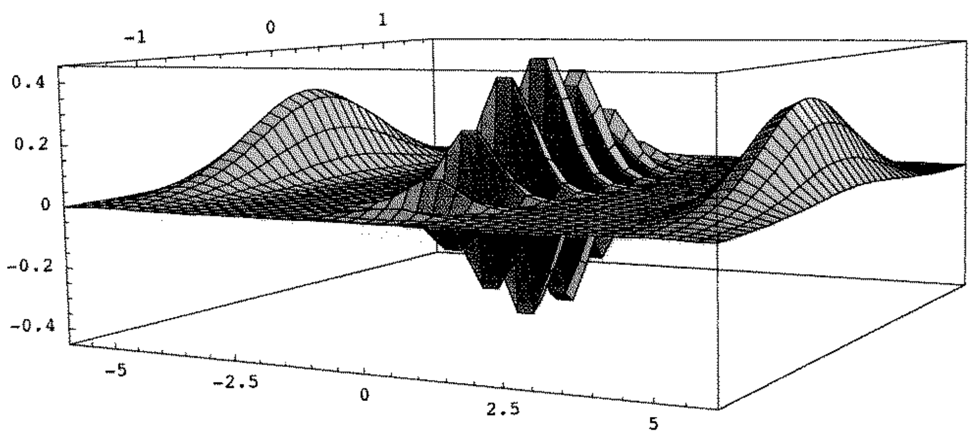
Note that as $\alpha_0 \to \infty$ (very nonclassically) the oscillations get more and more rapid.

This picture shows a kind of "interference in phase space". It captures both the position and momentum representations.

```
In[18]:= A = 2 e^(-2 x0^2) (1 + e^(2 x0^2))

Out[18]= 2 e^(-2 x0^2) (1 + e^(2 x0^2))

In[14]:= W[x_, p_] := 2 / (A Pi) (Exp[-2 ( (x - x0) ^2 + p^2)] +
           Exp[-2 ( (x + x0) ^2 + p^2)] + 2 Exp[-2 (x^2 + p^2)] Cos[4 x0 p])

In[84]:= x0 = 2;
         Plot3D[W[x, p], {x, -3, 3}, {p, -1.5, 1.5}, PlotRange → All, PlotPoints → 40,
          ViewPoint -> {1.409, -2.602, 0.320}]
```

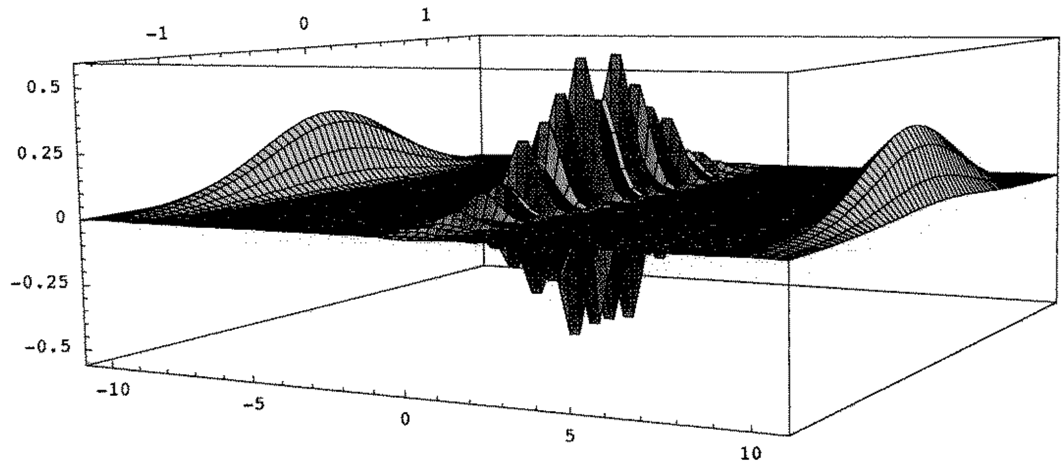



```
Out[85]= - SurfaceGraphics -

In[88]:= x0 = 5;
         Plot3D[W[x, p], {x, -6, 6}, {p, -1.5, 1.5}, PlotRange → All, PlotPoints → 40,
          ViewPoint -> {1.409, -2.602, 0.320}]
```




```
Out[89]= - SurfaceGraphics -
```

```
In[100]:= x0 = 10;
          Plot3D[W[x, p], {x, -11, 11}, {p, -1.5, 1.5}, PlotRange → All, PlotPoints → 70,
           ViewPoint -> {1.409, -2.602, 0.320}]
```

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Out[101]= - SurfaceGraphics -
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Problem 4: thermal Light

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}} \qquad Z = \mathrm{Tr}(e^{-\beta \hat{H}})$$

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a} \qquad \Rightarrow \quad Z = \sum_{n=0}^{\infty} (e^{-\beta\hbar\omega})^n = \frac{1}{1-e^{-\beta\hbar\omega}}$$

(a) $$\langle \hat{n} \rangle = \frac{1}{Z}\sum_{n=0}^{\infty} n (e^{-\beta\hbar\omega})^n = \frac{1}{Z} \frac{d}{d(-\beta\hbar\omega)} \sum_{n=0}^{\infty} (e^{-\beta\hbar\omega})^n$$

$$= \frac{1}{Z}\left(\frac{-e^{-\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2}\right) = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega}-1}$$

$$P_n = \frac{e^{-n\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^{-1}} \qquad \text{now} \quad e^{\beta\hbar\omega} = 1 + \frac{1}{\langle n \rangle} = \frac{\langle n \rangle + 1}{\langle n \rangle}$$

$$\Rightarrow P_n = \frac{\langle n \rangle^n}{(\langle n \rangle + 1)^n} \left[\frac{1}{1 - \frac{\langle n \rangle}{\langle n \rangle + 1}}\right]^{-1} = \frac{\langle n \rangle^n}{(1+\langle n \rangle)^{n+1}}$$

$$\Delta n^2 = \langle n^2 \rangle + \langle n \rangle^2$$

$$\langle \Delta n^2 \rangle = -\frac{d}{d(\beta\hbar\omega)} \langle \bar{n} \rangle \qquad (\text{see e.g. Reif})$$

$$= \frac{+e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2} = \frac{e^{\beta\hbar\omega} - 1 + 1}{(e^{\beta\hbar\omega}-1)^2}$$

$$= \frac{1}{e^{\beta\hbar\omega}-1} + \frac{1}{(e^{\beta\hbar\omega}-1)^2}$$

$$= \langle n \rangle + \langle n^2 \rangle$$

fluctuations beyond coherent state

$$\langle \hat{a} \rangle = \mathrm{Tr}(\hat{\rho}\,\hat{a}) = \sum_n \langle n|\hat{a}|n\rangle \rho_{nn} = 0$$

$\Rightarrow$ No mean amplitude in constrast to coherent state $|\alpha\rangle$.

(b) Calculating the quasi-distribution

The simplest is the Q-function (Husimi distribution)

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle = \frac{1}{Z\pi}\langle\alpha|e^{-\beta\hat{H}}|\alpha\rangle$$

$$= \frac{1}{Z\pi}\langle\alpha|\, e^{-\beta\hbar\omega\hat{a}^\dagger\hat{a}}\,|\alpha\rangle$$

Aside: Here's a good trick to remember

$$\langle\alpha|e^{-i\theta\hat{a}^\dagger\hat{a}}|\alpha\rangle = \langle\alpha|\alpha e^{-i\theta}\rangle$$

$$= \exp\{-|\alpha - \alpha e^{i\theta}|^2/2\}\,\exp\{\underline{|\alpha|^2 e^{-i\theta} - |\alpha|^2 e^{+i\theta}}_{2}\}$$

$$= \exp\{-|\alpha|^2(1-e^{-i\theta})\}$$

This analytically continues $\theta \Rightarrow -i\beta\hbar\omega$

$$\Rightarrow \langle\alpha|\, e^{-\beta\hbar\omega\hat{a}^\dagger\hat{a}}\,|\alpha\rangle = \exp\{-|\alpha|^2\underbrace{(1-e^{-\beta\hbar\omega})}_{=Z^{-1}}\}$$

$$\therefore \boxed{Q(\alpha) = \frac{1}{\pi Z}\, e^{-|\alpha|^2/Z}}$$

Given the Q-function, we can find P and W using the relationship between their characteristic fncts.

$$\chi_Q(\beta) = \int d\alpha\, Q(\alpha)\, e^{\beta\alpha^* - \beta^*\alpha} = e^{-Z|\beta|^2}$$

(using Problem 1a)

$$\chi_Q(\beta) = \mathrm{Tr}(\hat{\rho}\,\hat{D}(\beta))\, e^{-|\beta|^2/2}, \quad \chi_W(\beta) = \mathrm{Tr}(\hat{\rho}\,\hat{D}(\beta))$$

$$\chi_P(\beta) = \mathrm{Tr}(\hat{\rho}\,\hat{D}(\beta))\, e^{+|\beta|^2/2}$$

$$\Rightarrow \chi_W(\beta) = e^{-(Z-\frac{1}{2})|\beta|^2} \qquad \chi_P(\beta) = e^{-(Z-1)|\beta|^2}$$

Aside $Z - 1 = \dfrac{1}{1-e^{-\beta\hbar\omega}} - 1 = \dfrac{1}{e^{\beta\hbar\omega}-1} = \langle n\rangle$

$$\Rightarrow \chi_W(\beta) = e^{-(\langle n\rangle + \frac{1}{2})|\beta|^2} \qquad \chi_P(\beta) = e^{-\langle n\rangle|\beta|^2}$$

$$\therefore \boxed{\begin{aligned} W(\alpha) &= \frac{1}{\pi(\langle n\rangle + \frac{1}{2})}\, e^{-|\alpha|^2/(\langle n\rangle+\frac{1}{2})} \\ P(\alpha) &= \frac{1}{\pi\langle n\rangle}\, e^{-|\alpha|^2/\langle n\rangle} \\ Q(\alpha) &= \frac{1}{\pi(\langle n\rangle+1)}\, e^{-|\alpha|^2/(\langle n\rangle+1)} \end{aligned}}$$

All Gaussian functions, with slightly different widths.

In the limit $\langle n \rangle \to 0$

$W(\alpha) \to \frac{2}{\pi} e^{-2|\alpha|^2}$ = Wigner function for a vacuum

$P(\alpha) \to \delta^{(2)}(\alpha)$ = P function for vacuum

$Q(\alpha) \to \frac{1}{\pi} e^{-|\alpha|^2} = \frac{1}{\pi} |\langle 0|\alpha\rangle|^2$ = Q function for vacuum

✓

In the view of quasiprobability the thermal state and vacuum are very similar. They are both Gaussian states centered at the origin. The width of the Gaussian depends on the "temperature"; i.e. $\langle n \rangle$, and the particular operator ordering used. The Q and W functions include quantum fluctuations; their widths never go to zero.

The P function describes the state as a statistical mixture of coherent states. At zero temperature ($\langle n \rangle = 0$) this distribution collapses to a delta function.

(C) To calculate $\Delta X_1^2(\theta)$ and $\Delta X_2^2(\theta)$ we use the Wigner function (Problem 2). These variances are just the variance of the Wigner Gaussian (independent of θ)

$$\boxed{\Delta X_1^2 = \Delta X_2^2 = \langle n \rangle + \frac{1}{2} \rightarrow \frac{1}{2} \text{ for } \langle n \rangle = 0}$$

To calculate $\Delta n^2 = \langle n^2 \rangle - \langle n \rangle^2$

Recall $\langle n^2 \rangle = \langle a^\dagger a a^\dagger a \rangle = \langle a^{\dagger 2} a^2 \rangle + \langle n \rangle$

$$\Rightarrow \Delta n^2 = \langle : n^2 : \rangle - \langle : n : \rangle^2 + \langle n \rangle$$

$$= \underbrace{\left(\Delta |\alpha|^2\right)^2}_{\Uparrow} + \langle n \rangle \leftarrow \text{"shot noise"}$$

Uncertainty in coherent state intensity from P-function

$$\Rightarrow \boxed{\Delta n^2 = \langle n \rangle^2 + \langle n \rangle}$$ Photon bunching!

The fluctuations in Δn^2 have two components

- "Shot noise" arising from the particle nature
- Statistical fluctuation in $|\alpha|^2$ for a finite temperature.