Physics 566: Quantum Optics

Problem Set 6: Solutions

Problem 1: Nonclassical light via the Kerr effect

The Optical Kerr effect involves an intensity dependent index of refraction

$$\Delta n = n_2 I \qquad n_2 \propto \chi^{(3)}$$

The induced polarization $P \propto \Delta n E$

$\Rightarrow$ Interaction energy $H \propto P E^* = \Delta n I^2$

Quantizing $\Rightarrow$ Normal order $H = \Delta n :I^2:$

(in appropriate units)

We take the quantum Hamiltonian

$$\hat{H} = \frac{\hbar \chi^{(3)}}{2} \hat{a}^{\dagger 2} \hat{a}^2$$

(a) For a strong input we can linearize

$\Longrightarrow$ Take $\hat{a} = \alpha + \hat{b}$ and keep terms only to quadratic order in $\hat{b}$

$$\Longrightarrow \hat{H} = \frac{\hbar \chi^{(3)}}{2} (\alpha^* + \hat{b}^\dagger)^2 (\alpha + \hat{b})^2 \simeq \frac{\hbar \chi^{(3)}}{2} (\alpha^{*2} + 2\hat{b}^\dagger \alpha^* + \hat{b}^{\dagger 2}) \times (\alpha^2 + 2\hat{b}\alpha + \hat{b}^2)$$

$$\simeq \frac{\hbar \chi^{(3)}}{2} \Big[|\alpha|^4 + 2|\alpha|^2 (\alpha \hat{b}^\dagger + \alpha^* \hat{b}) + 4|\alpha|^2 \hat{b}^\dagger \hat{b} + \left(\alpha^2 \hat{b}^{\dagger 2} + (\alpha^*)^2 \hat{b}^2 \right) \Big]$$

The Hamiltonian exhibits ~~three~~ four terms

- $\frac{\hbar\chi^{(3)}}{2}|\alpha|^4$: Constant, classical Kerr effect

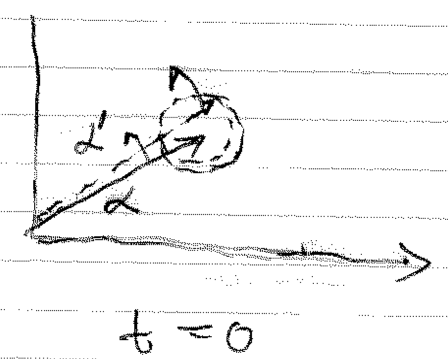
- $\hbar\,\chi^{(3)}|\alpha|^2(\alpha\hat{b}^\dagger + \alpha^*\hat{b})$ Phase-space displacement (parametric

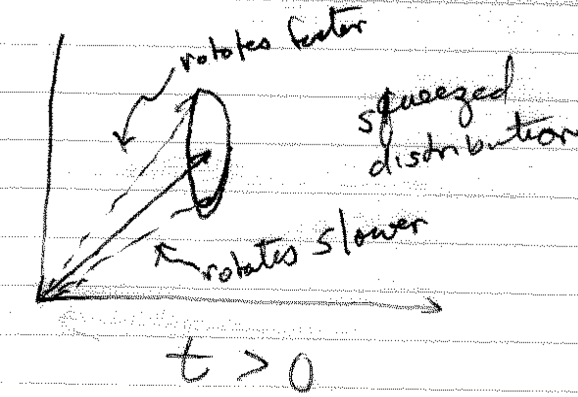
- $2\hbar\,\chi^{(3)}|\alpha|^2\,\hat{b}^\dagger\hat{b}$ Cross-phase modulation between pump and fluctuations $\Rightarrow$ rotation in phase space

- $\frac{\hbar\chi^{(3)}}{2}|\alpha|^2(\hat{b}^{\dagger 2}e^{2i\phi} + \hat{b}^2 e^{-2i\phi})$: Squeezing <u>Hamiltonian</u>

Thus, to this order we see that the Kerr effect will lead to squeezing. Because of the cross-phase modulation the axis of squeezing will continually rotate.

A simple classical picture explains this. Due to the Kerr effect, a phasor will rotate <u>faster</u> the longer the vector (bigger in density)





(b) The "shearing" of the uncertainty bubble eventually distorts it beyond elliptical. At this point, the linear approximation breaks down.

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$$

$$|\psi(0)\rangle = |\alpha\rangle = \sum_n c_n |n\rangle \qquad c_n = \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}$$

$$\hat{H} = \frac{\hbar\chi^{(3)}}{2}\hat{a}^{\dagger 2}\hat{a}^2 = \frac{\hbar\chi^{(3)}}{2}\left[(\hat{a}^\dagger\hat{a})^2 - \hat{a}^\dagger\hat{a}\right]$$

$$= \frac{\hbar\chi^{(3)}}{2}(\hat{n}^2 - \hat{n})$$

$$\therefore |\psi(t)\rangle = \sum_n e^{-i\frac{\chi^{(3)}}{2}t(n^2-n)} c_n|n\rangle$$

when $\chi^{(3)}t = \pi$

$$|\psi(t)\rangle = \sum_n e^{-i\frac{\pi}{2}(n^2-n)} c_n|n\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_n (-i)^{n^2} \frac{(+i\alpha)^n}{\sqrt{n!}}|n\rangle$$

Now $(-i)^{n^2} = \begin{cases} 1 & n \text{ even} \\ -i & n \text{ odd} \end{cases} = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4} + (-1)^n e^{i\pi/4}\right)$

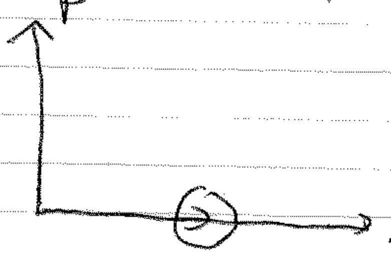
$$\Rightarrow |\psi(t)\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}} e^{-|\alpha|^2/2} \sum_n \frac{(i\alpha)^n}{\sqrt{n!}}|n\rangle$$

$$+ \frac{e^{i\pi/4}}{\sqrt{2}} e^{-|\alpha|^2/2} \sum_n \frac{(-i\alpha)^n}{\sqrt{n!}}|n\rangle$$

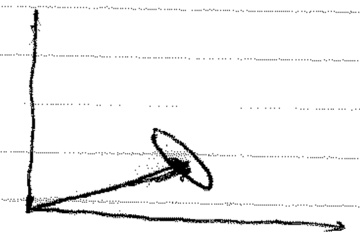
$$\Rightarrow \boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}\,|i\alpha\rangle + e^{i\pi/4}\,|-i\alpha\rangle\right)}$$

This is a so-called "Schrödinger cat state".

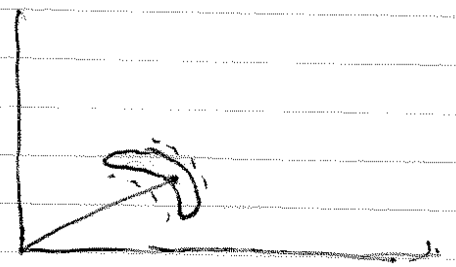
In phase space:



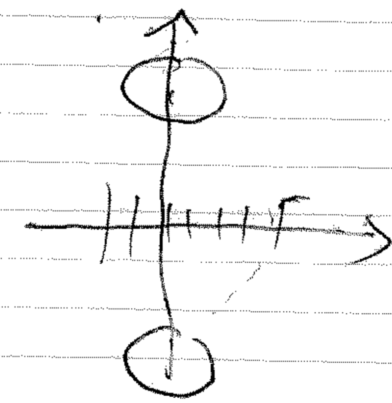
$t=0$ (take α real)



t small (squeezed)



t larger
(distorted ellipse)
(negative Wigner function)



Schrodinger cat!