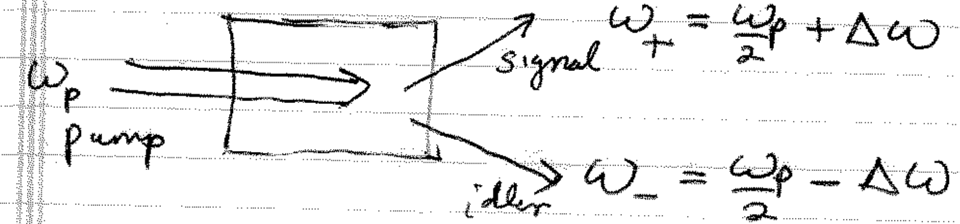
Problem 2: Two mode squeezed states



"Twin beams"
Quantum correlated

$$\hat{H} = i\hbar G\left(\hat{a}_+^\dagger \hat{a}_-^\dagger e^{-i\phi} - \hat{a}_+ \hat{a}_- e^{i\phi}\right)$$

Time evolution operator: $\hat{U}(t) = e^{-i\hat{H}t}$

$$\hat{U}(t) = e^{\xi \hat{a}_+ \hat{a}_- - \xi^* \hat{a}_+^\dagger \hat{a}_-^\dagger} = \hat{S}(\xi)$$

$$\xi = r e^{i\phi} \qquad r = Gt$$

(a) Generalized Bogoliubov transformation

$$\hat{S}^\dagger(\xi)\, \hat{a}_\pm\, \hat{S}(\xi) = \hat{a}_\pm + \frac{it}{\hbar}\left[\hat{H}, \hat{a}_\pm\right] + \frac{1}{2}\left(\frac{it}{\hbar}\right)^2 \left[\hat{H}, [\hat{H}, \hat{a}_\pm]\right] + \cdots$$

(Baker-Campbell-Hausdorff)

Aside: $\left[\hat{H}, \hat{a}_\pm\right] = -i\hbar G\, \hat{a}_\mp^\dagger\, e^{-i\phi}$

$$\left[\hat{H}, [\hat{H}, \hat{a}_\pm]\right] = -i\hbar G e^{-i\phi}\left[\hat{H}, a_\mp^\dagger\right]$$

$$= \left(-i\hbar G e^{-i\phi}\right)\left(i\hbar G e^{i\phi}\right)\hat{a}_\pm$$

$$= -(\hbar G)^2\, \hat{a}_\pm$$

etc.

$$\therefore\ \hat{S}^\dagger(\xi)\,\hat{a}_\pm\,\hat{S}(\xi) = \left(\sum_{n\,\text{even}} \frac{r^n}{n!}\right)\hat{a}_\pm - e^{-i\phi}\left(\sum_{n\,\text{odd}} \frac{r^n}{n!}\right)\hat{a}^\dagger_\mp$$

$$\Rightarrow \boxed{\hat{S}^\dagger(\xi)\,\hat{a}_\pm\,\hat{S}(\xi) = \cosh(r)\,\hat{a}_\pm - e^{-i\phi}\sinh(r)\,\hat{a}^\dagger_\mp}$$

(b) Consider quadrature operators:

$$\hat{X}_\pm \equiv \frac{\hat{a}_\pm e^{+i\theta} + \hat{a}^\dagger_\pm e^{-i\theta}}{2} \quad \text{for each mode}$$

$$\hat{S}^\dagger \hat{X}_\pm(\theta)\hat{S} = \frac{\left(c\hat{a}_\pm - e^{-i\phi} s\hat{a}^\dagger_\mp\right)e^{+i\theta} + \left(c\hat{a}^\dagger_\pm - e^{i\phi} s\hat{a}_\mp\right)e^{-i\theta}}{2}$$

$$= c\left(\frac{\hat{a}_\pm e^{+i\theta} + \hat{a}^\dagger_\pm e^{-i\theta}}{2}\right) - s\left(\frac{\hat{a}_\mp e^{-i(\theta-\phi)} \pm \hat{a}^\dagger_\mp e^{+i(\theta-\phi)}}{2}\right)$$

$$= c\,\hat{X}_\pm(\theta) - s\,\hat{X}_\mp(\theta-\phi) \qquad \begin{pmatrix} c \equiv \cosh r \\ s \equiv \sinh r \end{pmatrix}$$

Thus if we look at the fluctuations in these quadratures

$$\Delta\hat{X}_\pm^2 = \langle \hat{X}_\pm^2\rangle - \langle\hat{X}_\pm\rangle^2$$

(where the expectation value is taken in squeezed vacuum)

we will find no reduction below shot noise

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$$|0_\xi\rangle \equiv \hat{S}(\xi)\,|0\rangle_+ \otimes |0\rangle_-$$

$$\Rightarrow \langle 0_\xi|\,\hat{X}_\pm\,|0_\xi\rangle = 0$$

$$\langle 0_\xi|\,\hat{X}^2_\pm\,|0_\xi\rangle = {}_+\langle 0| \otimes {}_-\langle 0|\,\left(c\hat{X}_\pm(\theta) - s\,\hat{X}_\mp(\theta-\phi)\right)^2\,|0\rangle_+ \otimes |0\rangle_-$$

$$= c^2 \underbrace{\langle 0_+ 0_-|\,\hat{X}^2_\pm(\theta)\,|0_+ 0_-\rangle}_{\frac{1}{4}} + s^2 \underbrace{\langle 0_+ 0_-|\,\hat{X}^2_\mp(\theta-\phi)\,|0_+ 0_-\rangle}_{\frac{1}{4}}$$

$$= \frac{1}{4}\left(\cosh^2(r) + \sinh^2(r)\right) > \frac{1}{4}$$

$\Rightarrow$ No squeezing in single beams
(in fact extra noise in each beam)

Now consider the <u>noise difference</u>

$$\hat{Y}(\theta,\theta') \equiv \hat{X}_+(\theta) - \hat{X}_-(\theta')$$

$$\Rightarrow \hat{S}^\dagger\,\hat{Y}(\theta,\theta')\,\hat{S} = \left(c\,\hat{X}_+(\theta) - s\,\hat{X}_-(\theta-\phi)\right) - \left(c\,\hat{X}_-(\theta') - s\,\hat{X}_+(\theta'-\phi)\right)$$

$$= \left[c\,\hat{X}_+(\theta) + s\,\hat{X}_+(\theta'-\phi)\right]$$

$$- \left[c\,\hat{X}_-(\theta') + s\,\hat{X}_-(\theta-\phi)\right]$$

$$\langle \Delta \hat{Y}^2(\theta,\theta')\rangle = \langle (c\hat{X}_+(\theta) + s\hat{X}_+(\theta'-\phi))^2\rangle$$
$$+ \langle (c\hat{X}_-(\theta) + s\hat{X}_-(\theta'-\phi))^2\rangle$$

Aside:

$$\langle (c\hat{X}_+(\theta) + s\hat{X}_+(\theta'-\phi))^2\rangle$$
$$= c^2\langle \hat{X}_+^2(\theta)\rangle + s^2\langle \hat{X}_+^2(\theta'-\phi)\rangle$$
$$+ cs\left(\langle \hat{X}_+(\theta)\hat{X}_+(\theta'-\phi)\rangle + \langle \hat{X}_+(\theta'-\phi)\hat{X}_+(\theta)\rangle\right)$$

$$\langle \hat{X}_+^2(\theta)\rangle = \langle \hat{X}_+^2(\theta'-\phi)\rangle = \frac{1}{4}$$

$$\langle \hat{X}_+(\theta)\hat{X}_+(\theta'-\phi)\rangle = \left\langle \left(\frac{\hat{a}_+ e^{i\theta} + \hat{a}_+^\dagger e^{-i\theta}}{2}\right)\left(\frac{\hat{a}_+ e^{i(\theta'-\phi)} + \hat{a}_+^\dagger e^{-i(\theta'-\phi)}}{2}\right)\right\rangle$$

$$= \frac{1}{4}\langle \hat{a}_+\hat{a}_+^\dagger\rangle e^{i(\theta-\theta'-\phi)}$$

$$\therefore = \frac{c^2 + s^2 + 2cs\cos(\theta-\theta'-\phi)}{4}$$

$$= \frac{1}{4}\left[e^{-2r}\sin^2\left(\frac{\theta-\theta'-\phi}{2}\right) + e^{2r}\cos^2\left(\frac{\theta-\theta'-\phi}{2}\right)\right]$$

$$\therefore \boxed{\langle \Delta \hat{Y}^2(\theta,\theta')\rangle = \frac{1}{2}\left[e^{-2r}\sin^2\left(\frac{\theta-\theta'-\phi}{2}\right) + e^{2r}\cos^2\left(\frac{\theta-\theta'-\phi}{2}\right)\right]}$$

Interpretation:

When $\theta - \theta' - \phi = \pi$, the modes are maximally correlated $\Rightarrow$ Squeezing

(c) Number correlations in twin beams:

$$\hat{S}_{\pm}(r)\,|0_+\rangle|0_-\rangle = e^{r(\hat{a}_+^\dagger \hat{a}_-^\dagger - \hat{a}_+\hat{a}_+)}\,|0_+\rangle\otimes|0_-\rangle$$

take ξ real

To apply this operator, we need the disentangling theorem (see Walls + Milburn, eq. (5.63))

$$\hat{S}_{\pm}(r) = e^{\Gamma \hat{a}_+^\dagger \hat{a}_-^\dagger}\, e^{-g(\hat{a}_+^\dagger\hat{a}_+ + \hat{a}_-^\dagger\hat{a}_- + 1)}\, e^{-\Gamma \hat{a}_+\hat{a}_-}$$

where $\Gamma = \tanh(r)$ $\quad g = \ln(\cosh r)$

Applying this to the vacuum

$$\hat{S}_{\pm}(r)\,|0_+\rangle\otimes|0_-\rangle = e^{\Gamma \hat{a}_+^\dagger \hat{a}_-^\dagger}\, e^{-g}\,|0_+\rangle\otimes|0_-\rangle$$

$$= \frac{1}{e^g}\sum_{n} \Gamma^n \frac{(\hat{a}_+^\dagger)^n}{\sqrt{n!}}\frac{(\hat{a}_-^\dagger)^n}{\sqrt{n!}}\,|0_+\rangle\otimes|0_-\rangle$$

$$\Rightarrow \boxed{\hat{S}_{\pm}(r)\,|0_+\rangle\otimes|0_-\rangle = \frac{1}{\cosh(r)}\sum_{n}[\tanh(r)]^n\,|n_+\rangle|n_-\rangle}$$

Note for r small, we get the correlated photon pair (plus vacuum) d)

To obtain the margin density operator we trace over one of the degrees of freedom, e.g.,

$$\hat{\rho}_+ = \mathrm{Tr}_-\left(|\Psi_{+-}\rangle\langle\Psi_{+-}|\right)$$

$$= \frac{1}{\cosh^2(r)} \sum_n{}' \left[\tanh^2(r)\right]^n \, |n\rangle_+\langle n|$$

this has the form of a thermal state

$$\hat{\rho}_{thermal} = \sum_{n=0}^{\infty} P_n \, |n\rangle\langle n|$$

where $P_n = \dfrac{\left(e^{-\beta\hbar\omega}\right)^n}{Z} = \dfrac{\bar{n}^n}{(1+\bar{n})^{n+1}}$ is the Bose-Einstein distribution

setting $\left(\dfrac{\bar{n}}{1+\bar{n}}\right)^n \dfrac{1}{1+\bar{n}} = \dfrac{\left(\tanh^2(r)\right)^n}{\cosh^2(r)}$

using $\sinh^2(r) - \cosh^2(r) = 1$

$\tanh^2(r) - 1 = \dfrac{1}{\cosh^2(r)}$

$$\Longrightarrow \quad \boxed{\bar{n} = \sinh^2(r)}$$

(c) Wigner function of the two mode squeezed state

For two modes

$$W(\alpha_+, \alpha_-) = \frac{1}{\pi^4} \int d^2\beta_+ \int d^2\beta_- \, \chi(\beta_+, \beta_-) \, e^{\alpha_+ \beta_+^* - \alpha_+^* \beta_+}$$

$$\times \, e^{\alpha_- \beta_-^* - \alpha_-^* \beta_-}$$

where the characteristic function is

$$\chi(\beta_+, \beta_-) = \mathrm{Tr}\left(\hat{\rho} \, \hat{D}_+(\beta_+) \, \hat{D}_-(\beta_-)\right)$$

$$= \langle 0_r | \hat{D}_+(\beta_+) \, \hat{D}_-(\beta_-) | 0_r \rangle$$

$$|0_r\rangle = \exp\{ r(\hat{a}_+ \hat{a}_- - \hat{a}_+^\dagger \hat{a}_-^\dagger) \} |0_+\rangle \otimes |0_-\rangle$$

$$\Rightarrow \chi(\beta_+, \beta_-) = \langle 0_r | e^{\beta_+ \hat{a}_+^\dagger - \beta_+^* \hat{a}_+} \, e^{\beta_- \hat{a}_-^\dagger - \beta_-^* \hat{a}_-} | 0_r \rangle$$

$$= \langle 0_r | e^{(c\beta_+ - s\beta_-^*)\hat{a}_+^\dagger - (c\beta_+^* - s\beta_-)\hat{a}_+}$$

$$e^{(c\beta_- - s\beta_+^*)\hat{a}_-^\dagger - (c\beta_-^* - s\beta_+)\hat{a}_-} | 0_r \rangle$$

$$= \langle 0_+ | \hat{D}_+(c\beta_+ - s\beta_-^*) | 0_+ \rangle \langle 0_- | \hat{D}_-(c\beta_- - s\beta_+^*) | 0_- \rangle$$

$$= e^{-\frac{1}{2}|c\beta_+ - s\beta_-^*|^2} \, e^{-\frac{1}{2}|c\beta_- - s\beta_+^*|^2}$$

Fourier transforming, using our Gaussian integral from PS#X

$$W(\alpha_+, \alpha_-) = \frac{4}{\pi^2} \exp\left[-2|\alpha_+ \cosh r - \alpha_-^* \sinh r|^2 \right.$$

$$\left. -2|\alpha_- \cosh r - \alpha_+^* \sinh r|^2\right]$$

In terms of the quadrature $x_\pm + i p_\pm \equiv \alpha_\pm$

$$W(x_\pm, p_\pm) = \frac{4}{\pi^2} \exp\Big[-2\,(x_+ c - x_- s)^2 - 2\,(p_+ c + p_- s)^2 - 2\,(x_- c - x_+ s)^2 - 2\,(p_- c + p_+ s)^2\Big]$$

$$= \frac{4}{\pi^2} \exp\Big[-2(x_+^2 + x_-^2 + p_+^2 + p_-^2)(c^2 + s^2) + 8(x_+ x_- - p_+ p_-)\, cs\Big]$$

with $c^2 + s^2 = \dfrac{e^{2r} + e^{-2r}}{2}$ $\qquad cs = \dfrac{e^{2r} - e^{-2r}}{4}$

$$\Rightarrow W(x_\pm, p_\pm) = \frac{4}{\pi^2} \exp\Big[-e^{+2r}\left(x_+^2 + x_-^2 - 2x_+x_- + p_+^2 + p_-^2 + 2p_+p_-\right) - e^{-2r}\left(x_+^2 + x_-^2 + 2x_+x_- + p_+^2 + p_-^2 - 2p_+p_-\right)\Big]$$

$$\Rightarrow \boxed{W(x_\pm, p_\pm) = \frac{4}{\pi^2} \exp\Big[-e^{-2r}\{(x_+ - x_-)^2 + (p_+ + p_-)^2\} - e^{+2r}\{(x_+ + x_-)^2 + (p_+ - p_-)^2\}\Big]}$$

In the limit $r \to \infty$ the squeezed direction goes to zero and the stretched direction to ∞, yielding a delta function

$$\boxed{W(x_\pm, p_\pm) \longrightarrow C\ \delta(x_+ + x_-)\ \delta(p_+ - p_-)}$$

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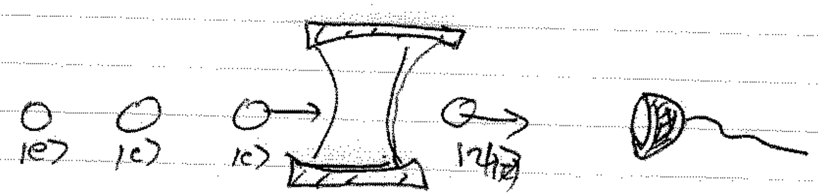
Problem 4: State preparation in cavity QED

In the classic Von Neumann paradigm of quantum measurement, a "system" is prepared by coupling it to a "meter" and then measuring the meter. We have a simple example of this in cavity QED. An atom can act as a meter for the field:



The unitary coupling a given atom and the mode was given in lecture

$$\hat{U}(t) = \cos(gt\sqrt{\hat{n}+1})\,|e\rangle\langle e| + \cos(gt\sqrt{\hat{n}})\,|g\rangle\langle g|$$
$$- i\,\frac{\sin(gt\sqrt{\hat{n}+1})}{\sqrt{\hat{n}+1}}\,\hat{a}\,|e\rangle\langle g| - i\,\hat{a}^\dagger\,\frac{\sin(gt\sqrt{\hat{n}+1})}{\sqrt{\hat{n}+1}}\,|g\rangle\langle e|$$

Initially the system is in state $|e\rangle\otimes|0\rangle$ (excited atom, field in vacuum)

➜ After first atom interacts for time t

$$|\psi(t)\rangle = \hat{U}(\tau)\,|e\rangle\otimes|0\rangle$$
$$= \cos(gt)\,|e\rangle|0\rangle - i\sin(gt)\,|g\rangle|1\rangle$$

⇒ If atom is found in $|e\rangle$, the cavity has zero photons (probability $\cos^2(gt)$)

If atom is found in $|g\rangle$, the cavity has one photon (probability $\sin^2(gt)$)

Let us now suppose $m-1$ atoms pass.

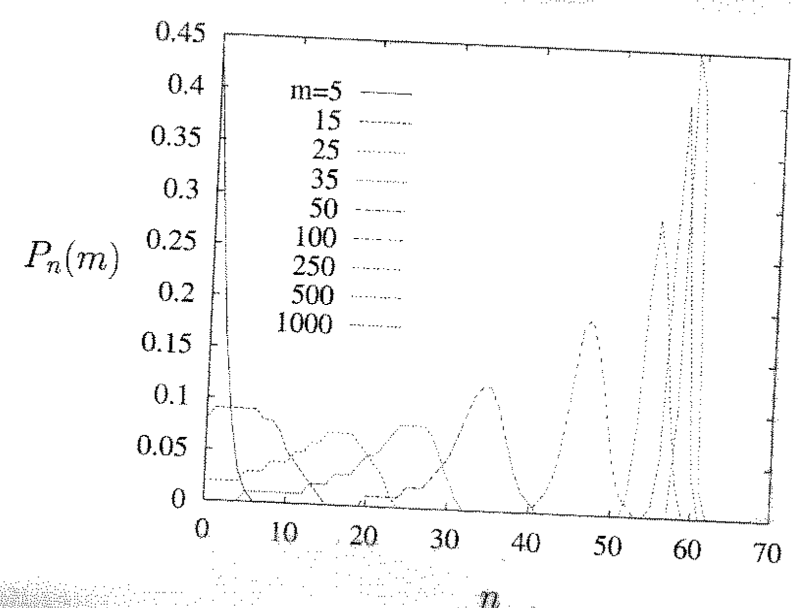
Let $P_n(m)$ = Probability of n photons after m atoms pass

$\Rightarrow$ If there are n photons after m atoms then after $m-1$ atoms there ~~the~~ was either n or $n-1$ photons (atom can emit or not)

$$P_n(m) = \underbrace{|\langle e n | U(\tau) | e n \rangle|^2}_{\cos^2(g t \sqrt{n+1})} P_n(m-1) + \underbrace{|\langle g n | U(\tau) | e\, n-1 \rangle|^2}_{\sin^2(g t \sqrt{n})} P_{n-1}(m-1)$$

(b) Using this recursion relation ~~with~~ with the initial condition $P_n(0) = \delta_{n0}$ we can solve numerically

Fig. 13.5 Probability of obtaining n photons in the cavity after m atoms have passed for $g\tau = 0.4$. (From J. Krause, M. O. Scully, and H. Walther, *Phys. Rev. A* **36**, 4547 (1987).)



As $m \to \infty$ the distribution becomes sharper.