Physics 566 Fall 2004

Problem Set #4

Due: Thursday Sept. 23, 2004

Problem 1: Free induction decay (10 Points)

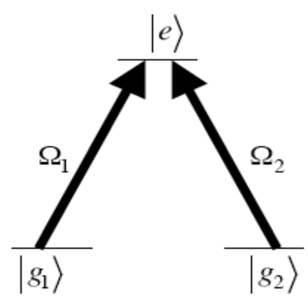
(a) Assume a 2-level atom, initially in the ground state. At t = 0, apply a pulse of light, with oscillation frequency wo, i.e., on resonance. The shape of the pulse is described by an envelope function E(t), with E(0) = E(T) = 0. Assume that $T << \Gamma^{-1}$, so you can ignore spontaneous emission. Find the condition on $\int dt\, E(t)$ from 0 to T such that after the pulse is over, $\rho_{ee} = 1/2$. What is ρ_{eg} (in the rotating frame)?

(b) Take the case of sodium where Γ^{-1} = 16 ns, and the optical wavelength is 589 nm. Take E(t) to be constant from t =0 to t =T, and let T = 100 ps (this is a typical pulse length for a garden variety mode-locked laser). Assume, as we have before, that the atom is a 2-level atom, with the transition dipole moment being along *z*, and that the laser is polarized along *z*. Calculate what intensity (in Watts/cm^2) is required to achieve the $\pi/2$ pulse of part (a). Compare this power to the power such that the Rabi frequency $\Omega = \Gamma$.

(c) Starting with the atom as in part (a), after the pulse, calculate the evolution of the density matrix (ρ_{ee} and ρ_{eg}). Describe the evolution. This is the free induction decay due to spontaneous emission.

Problem 2: Dark states (10 points)

Let us consider again a three level "lambda system"



The two ground states are resonantly coupled to the excited state, each with a different Rabi frequency. Taking the two ground states as the zero of energy, then in the RWA (and in the rotating frame) the Hamiltonian is

$$\hat{H}_{AL} = -\frac{\hbar}{2}\left[\Omega_1\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \Omega_2\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)\right]$$

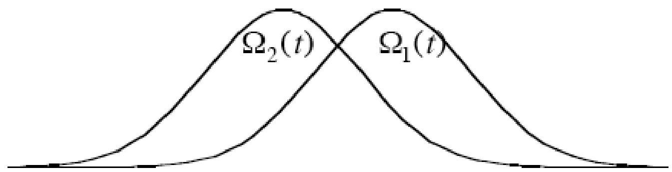
(a) Find the "dressed states" of this system (i.e. the eigenstates and eigenvalues of the total atom laser system). You should find that one of these states has a **zero** eigenvalue,

$$|Dark\rangle = \Omega_2|g_1\rangle - \Omega_1|g_2\rangle$$

This particular superposition is called a "**dark state**" or uncoupled state because the laser field does not couple it to the excited state. Explain how this can be true.

(b) Adiabatic transfer through the "nonintuitive" pulse sequence. Suppose we want to transfer population from $|g_1\rangle$ to $|g_2\rangle$. Could try to Raman-Rabi flop between these states

as in the previous problem set. An alternative, and robust method is to use adiabatic passage, *always staying in the local dark state*. This can then be *on resonance*.
Show that if we apply a slowly varying pulse $\Omega_2(t)$ overlapped, but followed by $\Omega_1(t)$ shown below,



we accomplish this transfer. **Hint**: Sketch the dressed state eigenvalues a function of time. Note, the pulse sequence going from $|g_1\rangle \rightarrow |g_2\rangle$ is "counter intuitive" as a realtransition involving absorption and emission. This is quantum mechanics!

Problem 3: Momentum and Angular Momentum in the E&M Field (20 points)
From classical electromagnetic field theory we know that conservation laws require that the field carry momentum and angular momentum

$$\mathbf{P} = \int d^3x \left(\frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right), \; \mathbf{J} = \int d^3x \left(\mathbf{x} \times \frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right).$$

(a) Show that when these quantities become field operators, the momentum operator becomes, $\hat{\mathbf{P}} = \sum_{\mathbf{k},\lambda} \hbar \mathbf{k} \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}$; interpret.

(b) Show that $\mathbf{J} = \mathbf{J}_{orb} + \mathbf{J}_{spin}$

where $\mathbf{J}_{orb} = \dfrac{1}{4\pi c} \int d^3x\, E_i(\mathbf{x}) (\mathbf{x} \times \nabla) A_i(\mathbf{x})$, $\mathbf{J}_{spin} = \dfrac{1}{4\pi c} \int d^3x\, (\mathbf{E}(\mathbf{x}) \times \mathbf{A}(\mathbf{x}))$

(c) Show that

$\hat{\mathbf{J}}_{orb} = \hbar \sum_{\mathbf{k},\lambda} \hat{a}^\dagger_{\mathbf{k},\lambda} (i\nabla_{\mathbf{k}} \times \mathbf{k}) \hat{a}_{\mathbf{k},\lambda}$, where $\nabla_{\mathbf{k}}$ is the gradient in **k-space**, and

$\hat{\mathbf{J}}_{spin} = \hbar \sum_{\mathbf{k}} \left(\hat{a}^\dagger_{\mathbf{k},+} \hat{a}_{\mathbf{k},+} - \hat{a}^\dagger_{\mathbf{k},-} \hat{a}_{\mathbf{k},-} \right) \mathbf{e}_{\mathbf{k}}$. Interpret these quantities.

(d) The spin of the photon has magnitude S=1, yet there are only two helicity states. Thus we can map the spin angular momentum onto the Bloch(Poincaré) sphere for S=1/2, via

$$\hat{\mathbf{J}}_{spin} = \hat{J}_x \mathbf{e}_x + \hat{J}_y \mathbf{e}_y + \hat{J}_z \mathbf{e}_z,$$

with $J_z = \dfrac{\hbar}{2} (\hat{a}^\dagger_{z+} \hat{a}_{z+} - \hat{a}^\dagger_{z-} \hat{a}_{z-})$, $J_x = \dfrac{\hbar}{2} (\hat{a}^\dagger_{z+} \hat{a}_{z-} + \hat{a}^\dagger_{z-} \hat{a}_{z+})$, $J_y = \dfrac{\hbar}{2i} (\hat{a}^\dagger_{z+} \hat{a}_{z-} - \hat{a}^\dagger_{z-} \hat{a}_{z+})$,

where $(\hat{a}_{z+}, \hat{a}_{z-})$ are the mode operators for positive and negative helicity operators relative to a *space fixed* quantization axis.

(di) Show that these operators satisfy the SU(2) commutation algebra for angular momentum. This relationship is know as the "Schwinger representation" (see Sakauri).

(dii) The mean values of $\hat{J}_x, \hat{J}_y, \hat{J}_z$ are the "Stokes parameters" in classical optics and the Bloch vector components on the Poincaré sphere. Explain the relationship between these operators and the Pauli operators.