Physics 566, Quantum Optics
Problem Set #8
Due: Thurs. Nov. 4, 2004

Problem 1: Twin beams and two-mode squeezed states. (15 points)
In lecture 14 we discussed how parametric downconversion leads to correlated twin "signal" and "idler" beams as long as the phase matching conditions are satisfied,

$$\omega_p = \omega_s + \omega_i, \;\; \mathbf{k}_p = \mathbf{k}_s + \mathbf{k}_i.$$

Considering two nondegenerate modes, $\omega_\pm = \omega_p/2 \pm \Delta\omega$, the Hamiltonian is

$$\hat{H} = i\hbar G\left(\hat{a}_+^\dagger \hat{a}_-^\dagger e^{-i\phi} - \hat{a}_+\hat{a}_- e^{i\phi}\right),$$

where G is the coupling constant depending on the nonlinearity, pump amplitude, and vacuum mode strength. The state produced is known as a "two-mode squeezed vacuum state",

$$\hat{S}_\pm(\xi)|0\rangle_+ \otimes |0\rangle_- = \exp\left[\frac{1}{2}\left(\xi\hat{a}_+\hat{a}_- - \xi^*\hat{a}_+^\dagger\hat{a}_-^\dagger\right)\right]|0\rangle_+ \otimes |0\rangle_-,$$

where $\xi = re^{i\phi}$ is the complex squeezing parameter for an interaction time t, $r = Gt$.

(a) Show that the generalized Bogoliubov transformations is

$$\hat{S}_\pm^\dagger(\xi)\hat{a}_\pm\hat{S}_\pm(\xi) = \cosh(r)\hat{a}_\pm - e^{i\phi}\sinh(r)\hat{a}_\mp^\dagger.$$

(b) Show that the individual modes, $\hat{a}_\pm$, show no squeezing, but that squeezing exists in the *correlation* between the modes. Hint: consider quadratures,

$$\hat{X}_\pm(\theta) \equiv \frac{\hat{a}_\pm e^{i\theta} + \hat{a}_\pm^\dagger e^{-i\theta}}{2} \text{ and then } \hat{Y}(\theta,\theta') \equiv \left(\hat{X}_+(\theta) - \hat{X}_-(\theta')\right)/\sqrt{2}.$$

For the remaining parts, take ξ real.

(c) The two-mode squeezed state is an entangled state between the signal and idler as we know from the perturbative limit of twin photons. Show that in the Fock basis

$$\hat{S}_\pm(r)|0\rangle_+ \otimes |0\rangle_- = \left(\cosh(r)\right)^{-1}\sum_{n=0}^{\infty}\left(\tanh(r)\right)^n |n\rangle_+ \otimes |n\rangle_-.$$

Hint: Use the "disentangling theorem" (D. R. Traux, Phys. Rev. D **31**, 1988 (1985)):

$$e^{r\left(\hat{a}_+^\dagger \hat{a}_-^\dagger - \hat{a}_+ \hat{a}_-\right)} = e^{\Gamma \hat{a}_+^\dagger \hat{a}_-^\dagger} e^{-g\left(\hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_- + 1\right)} e^{-\Gamma \hat{a}_+ \hat{a}_-}.$$

$$\text{where } \Gamma = \tanh(r),\ g = \ln\left(\cosh(r)\right)$$

The photons are produced with perfect correlations between the modes. This is known as "number squeezing".

(d) Show that the marginal density operator for each mode is a *thermal state* with mean photon number $\bar{n} = \sinh^2(r)$.

(e) This entangled state produced in twin beam generation is very close to the form considered by Einstein-Podolfsky-Rosen in their famous paradox.

Show that the Wigner function for our state in the two modes is,

$$\begin{aligned} W(\alpha_+,\alpha_-) &= \frac{4}{\pi^2}\exp\left\{-e^{-2r}\left[(x_+ - x_-)^2 + (p_+ + p_-)^2\right] - e^{+2r}\left[(x_+ + x_-)^2 + (p_+ - p_-)^2\right]\right\} \\ &\to C\delta(x_+ + x_-)\delta(p_+ - p_-) \end{aligned}$$

where $\alpha_\pm = x_\pm + ip_\pm$ the final expression is the limit of infinite squeezing, $r \to \infty$. For mechanical degrees of freedom, $x_\pm$ and $p_\pm$ represent position and momentum of two particles which are perfectly correlated. The quantum optical implementation maps these onto mode quadratures which are very tightly correlated. Violations of Bell's inequalties can then be measured (see Z.Y. Ou *et al.*, Phys. Rev Lett. **68**, 3663 (1992)).

(f) Extra credit: The Wigner function is positive, but clearly the EPR state clear has nonclassical features. Your thoughts?

Problem 3: Coupled Simple Harmonic Oscillators and Entangled States (15 Points)

Consider two simple Harmonic oscillators, each with a natural frequency ω, linearly coupled together. The Hamiltonian describing such a system can be written:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$
$$\hat{H}_0 = \hbar\omega\,\hat{a}^\dagger\hat{a} + \hbar\omega\hat{b}^\dagger\hat{b}, \quad H_{\text{int}} = \hbar\kappa\left(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\right),$$

where a and b are the annihilation operators for the two oscillators satisfying the commutation relations,

$$\left[\hat{a},\hat{a}^\dagger\right] = 1, \quad \left[\hat{b},\hat{b}^\dagger\right] = 1, \quad \left[\hat{a},\hat{b}\right] = [\hat{a},\hat{b}^\dagger] = 0,$$

and κ is the coupling constant.

Recall the "interaction picture", defined by performing the unitary transformation on the states and operators in the Schrödinger picture,

$$|\psi\rangle^{(I)} = \hat{U}_0^\dagger|\psi\rangle^{(S)}, \quad \hat{A}^{(I)} = \hat{U}_0^{\dagger}\,\hat{A}^{(S)}\,\hat{U}_0, \text{ with } \hat{U}_0 = \exp\left(\frac{-i}{\hbar}\hat{H}_0 t\right).$$

(a) Show that in the interaction picture, the state vector evolves according to

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle^{(I)} = \hat{H}_{\text{int}}^{(I)}|\psi\rangle^{(I)},$$

and for the Hamiltonian above, $\hat{H}_{\text{int}}^{(I)}$ is time independent so that, the state vector in the Schrödinger picture

$$|\psi(t)\rangle^{(S)} = \hat{U}_0\ \exp\left(\frac{-i}{\hbar}\hat{H}_{\text{int}}^{(I)}t\right)|\psi(0)\rangle^{(I)} \quad \text{(Note that } |\psi(0)\rangle^{(S)} = |\psi(0)\rangle^{(I)}\text{)}$$

(b) Show that in this picture, $\hat{a}(t) = \hat{a}\,\cos(\kappa t) - i\,\hat{b}\,\sin(\kappa t)$, where $\hat{a},\hat{b}$ are the Schrödinger picture operators.

(c) Suppose that the initial state (in either the Schrödinger or Interaction picture) has oscillator "a" in the first excited state, and oscillator "b" in the ground state:

$$|\psi(0)\rangle^{(S)} = |1\rangle_a \otimes |0\rangle_b = a^\dagger |0\rangle_a \otimes |0\rangle_b .$$

Show that the state vector for the total system at some later time t>0 is,

$$|\psi(t)\rangle^{(S)} = e^{-i\omega t}\cos(\kappa t)\,|1\rangle_a \otimes |0\rangle_b + i\,e^{-i\omega t}\sin(\kappa t)|0\rangle_a \otimes |1\rangle_b$$

Explain what this result means physically.

(d) Find the density matrix for the total system as a function of time. Is it a pure state?

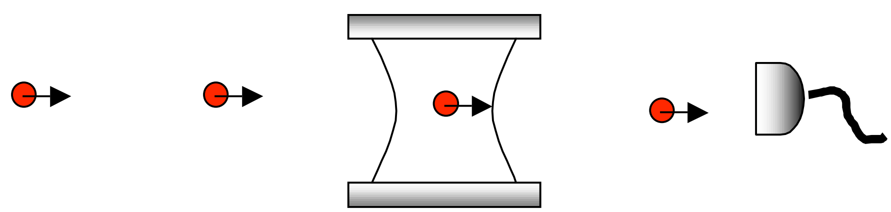
(e) Show that the *reduced* density matrix for the state of oscillator "a" alone is,

$$\rho^{(a)}_{reduced} = \cos^2(\kappa t)\,|1\rangle_a\langle 1|_a + \sin^2(\kappa t)|0\rangle_a\langle 0|_a$$

Is it normalized.

(f) Show that $\rho^{(a)}_{reduced}$ oscillates between a pure state and a mixed state with a frequency, $\Omega = 2\kappa$. Explain this result.

Problem 3: State preparation in cavity QED. (10 points)
In the Jaynes-Cummings system, the atom and field become entangled. By measuring the state of the atom at some time, we gain information about the state of the field. Consider the following geometry,



A stream of atoms, prepared in a very long lived excited state, are sent through a very high-Q cavity interacting with the mode for a time τ. After the atom emerges, it is measured and determined to be in either the excited or ground state. The cavity is initially prepared in the vacuum before any atoms enter.

(a) Show that after m atoms interact and are measured, the probability of having n photons in the cavity satisfies the recursion relation

$$P_n(m) = \cos^2\left(g\tau\sqrt{n+1}\right)P_n(m-1) + \sin^2\left(g\tau\sqrt{n}\right)P_{n-1}(m-1).$$

(b) Solve numerically, and plot as a function of n for m=5,25, 50, 100. Take $g\tau$=0.4. Comment.