Physics 566: Quantum Optics

Lecture 3: Magnetic Resonance and Rabi Flopping

All coherent laser spectroscopy has at its heart, spin magnetic resonance.

Father of the subject I. Rabi

1939 : measured hyperfine structure + ~~magnet~~ Lamb shift

The problem of manipulating spin-1/2

Hamiltonian, like any operator on 2D space

$$\hat{H} = A\hat{\mathbb{1}} + \vec{B}\cdot\hat{\vec{\sigma}}$$

A constant (defines zero of energy)

$\Rightarrow$ All Hamiltonians for 2D Hilbert space $\equiv$ Spin in a magnetic field

"Zeeman" Hamiltonian (static $\vec{B}_0$)

$$\hat{H}_0 = -\hat{\vec{\mu}}\cdot\vec{B}_0 = -\underbrace{\gamma}_{\text{gyro-magnetic ratio}}\hat{\vec{S}}\cdot\vec{B}_0 = -\frac{\hbar\gamma}{2}\hat{\vec{\sigma}}\cdot\vec{B}_0$$

$$\hat{H}_0 = -\frac{\hbar\vec{\Omega}_0}{2}\cdot\vec{\sigma} = -\frac{\hbar|\vec{\Omega}_0|}{2}\vec{e}_n\cdot\vec{\sigma}$$

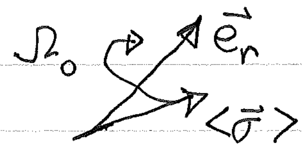
where $\gamma\vec{B}_0 = |\vec{\Omega}_0|\,\vec{e}_n$

$|\vec{\Omega}_0|$: Larmor frequency

Unitary evolution

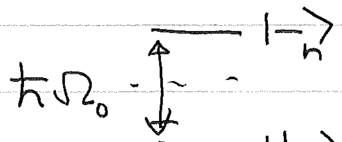
$$e^{-\frac{i}{\hbar}\hat{H}_0 t} = e^{i\Omega_0 t \frac{\sigma_n}{2}} = \text{rotation about } -\vec{e}_n \text{ by angle } \Omega_0 t$$

$\equiv$ Larmor precession



$\equiv$ Gyroscope

Eigenstates: $|\pm_n\rangle$ $\qquad H_0|\pm_n\rangle = \mp\frac{\hbar\Omega_0}{2}|\pm_n\rangle$



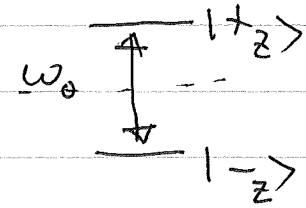
spin aligned along $\vec{e}_n$ has the lower energy where $\gamma > 0$

Magnetic resonance:

- Apply a strong static magnetic feed $\vec{B}_0$ along some axis (call it $-\vec{e}_z$ so $|-_z\rangle$ is the lowest energy state)

$$\vec{B}_0 \equiv -B_{\parallel}\vec{e}_z$$

$$\hat{H}_0 = -\vec{\mu}\cdot\vec{B}_0 \equiv \frac{\hbar\omega_0}{2}\hat{\sigma}_z \qquad \text{where } \omega_0 = \gamma B_{\parallel}$$



- "Drive" the system between $|\pm_z\rangle$ by applying an oscillation $\vec{B}$-field @ frequency ω, near to the resonant frequency ω_0.

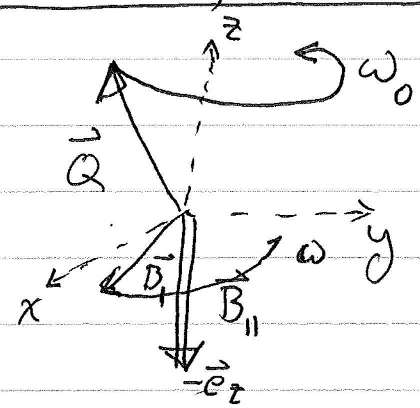
We thus have a perturbation Hamiltonian

$$\hat{H}_1(t) = -\vec{\mu} \cdot \vec{B}_1(t) \leftarrow \text{oscillating } \vec{B}\text{-field}$$

In order to achieve a non-trivial evolution, we require

$$[\hat{H}_1(t), \hat{H}_0] \neq 0 \Rightarrow \vec{B}_1(t) \text{ in } x\text{-}y \text{ plane ("transverse")}$$

<u>To achieve resonance, consider the following geometry</u>



In the absense of the perturbation, $\vec{Q}$ precesses clockwise about z-axis @ ω_0

By applying a small transverse B-field, $\vec{B}_1$, that rotates <u>with</u> $\vec{Q}$ ($\omega \approx \omega_0$), the in the rotating frame $\vec{Q}$ with be quasi-static. In that frame the spin will <u>flip</u>.

$\Rightarrow$ resonance!

Quantitatively

Choose: $\vec{B}_1 = B_\perp \left(\cos(\omega t + \phi)\, \vec{e}_x + \sin(\omega t + \phi)\, \vec{e}_y \right)$

arbitrary phase

$$\Rightarrow \hat{H}_1(t) = -\frac{\gamma \hbar B_\perp}{2} \left(\cos(\omega t + \phi)\, \hat{\sigma}_x + \sin(\omega t + \phi)\, \hat{\sigma}_y \right)$$

$$= -\frac{\hbar \Omega}{2} \left(\hat{\sigma}_+ e^{-i(\omega t + \phi)} + \hat{\sigma}_- e^{i(\omega t + \phi)} \right)$$

where $\Omega \equiv \gamma B_\perp \equiv$ Rabi frequency

Note: $\hat{\sigma}_+ = |+_z\rangle\langle -_z|$ "absorption" $e^{i\omega t}$

$\hat{\sigma}_- = |-_z\rangle\langle +_z|$ "emission" $e^{-i\omega t}$

Solve for evolution

Given $|\psi(0)\rangle = |-_z\rangle$, $\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t)$

find state @ a later time.

Solution #1: Time-dependent perturbation theory

Given $\hat{H}_1(t) = \hat{H}_{abs}\, e^{-i\omega t} + \hat{H}_{emiss}\, e^{+i\omega t}$

Fermi's Golden Rule

$$\text{Rate of absorption} = \frac{2\pi}{\hbar^2} \left| \langle +_z | \hat{H}_{abs} | -_z \rangle \right|^2 \mathcal{D}(\omega)$$

density of states

For our case:

$$R_{abs} = \frac{2\pi}{\hbar^2}\left(\frac{\hbar^2\Omega^2}{4}\right)\mathcal{D}(\omega) = \frac{\pi\Omega^2}{2}\mathcal{D}(\omega)$$

Fermi's Golden Rule $\Rightarrow$ "~~Ph~~ Incoherent" jump from $|-_z\rangle \Rightarrow |+_z\rangle$

This is not the whole story. It is applicable for

- Incoherent, broad-band source (e.g. lamp)

and/or

- Final state is "broad"

Isolated two-level system ~~coupled to~~ with quasi-monochromatic source can be solved beyond perturbation as coherent unitary evolution

Solution #2. Rabi flopping

Step #1 - Make the Hamiltonian time-independent by going to a "rotating frame", rotating at ω with $\vec{B}_\perp(t)$

Rotation is about $\vec{e}_z$: $\hat{D}(t) = e^{-i\omega t\frac{\hat{\sigma}_z}{2}}$

In the "new frame" we have new state and new operators (like interaction picture)

$$|\tilde{\Psi}\rangle \equiv \hat{D}^\dagger(t)|\Psi\rangle \quad \left(\begin{matrix}\text{Remove rotation}\\ \text{from spin}\end{matrix}\right)$$

$$\tilde{A}(t) \equiv \hat{D}^\dagger(t)\,\underset{\text{Schrödinger picture}}{\hat{A}(t)}\,\hat{D}(t)$$

Time evolution of state in "rotating frame"

$$i\hbar \frac{\partial}{\partial t} |\tilde{\Psi}(t)\rangle = i\hbar \frac{\partial}{\partial t} \hat{D}(t) |\Psi(t)\rangle$$

$$= \left\{ \hat{D}^\dagger(t) i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle + i\hbar \left(\frac{\partial \hat{D}^\dagger}{\partial t}\right) |\Psi(t)\rangle \right\}$$

$$= \hat{D}^\dagger(t) \hat{H}(t) |\Psi(t)\rangle + i\hbar \left(\frac{\partial \hat{D}^\dagger}{\partial t}\right) |\Psi(t)\rangle$$

$$= \underbrace{\left\{ \hat{D}^\dagger(t) \hat{H}(t) \hat{D}(t) + i\hbar \frac{\partial \hat{D}^\dagger(t)}{\partial t} \hat{D}(t) \right\}}_{\tilde{H} \,\leftarrow\, \text{new Hamiltonian}} |\tilde{\Psi}(t)\rangle$$

Aside
$$\left[\begin{array}{l} e^{i\omega t \frac{\hat{\sigma}_z}{2}} \hat{\sigma}_\pm e^{-i\omega t \frac{\hat{\sigma}_z}{2}} = \hat{\sigma}_\pm e^{\pm i\omega t} \\ i\hbar \frac{\partial \hat{D}(t)}{\partial t} \hat{D}^\dagger(t) = -\frac{\hbar\omega}{2} \hat{\sigma}_z \end{array}\right.$$

Thus
$$\tilde{H} = H_0 + i\hbar \frac{\partial \hat{D}}{\partial t} \hat{D}^\dagger + \tilde{H}_1$$

$$\tilde{H} = -\frac{\hbar(\omega-\omega_0)}{2} \hat{\sigma}_z - \frac{\hbar\Omega}{2} \left(\hat{\sigma}_+ e^{-i\phi} + \hat{\sigma}_- e^{i\phi}\right)$$

$$\Rightarrow \boxed{\begin{array}{l} \tilde{H} = -\frac{\hbar\Delta}{2} \hat{\sigma}_z - \frac{\hbar\Omega}{2} \left(\hat{\sigma}_x \cos\phi + \hat{\sigma}_y \sin\phi\right) \\ \Delta = \omega - \omega_0 \quad \text{"detuning"} \\ \Omega = \gamma B_\perp \quad \text{"Rabi frequency"} \end{array}}$$

Thus, in the rotating frame, the new Hamiltonian is <u>static</u> and of the form

$$\tilde{H} = -\frac{\hbar \vec{\Omega}_{eff}}{2} \cdot \vec{\sigma}$$

where $\vec{\Omega}_{eff} = \Delta \vec{e}_z + \Omega\, \vec{e}_\perp$ ← $\vec{e}_\perp = \cos\phi\, \vec{e}_x + \sin\phi\, \vec{e}_y$

"Generalized Rabi frequency"

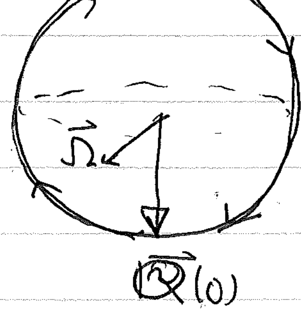
$$\Omega_{eff} \equiv |\vec{\Omega}_{eff}| = \sqrt{\Omega^2 + \Delta^2}$$

Axis of Larmor precession (Rabi flopping)

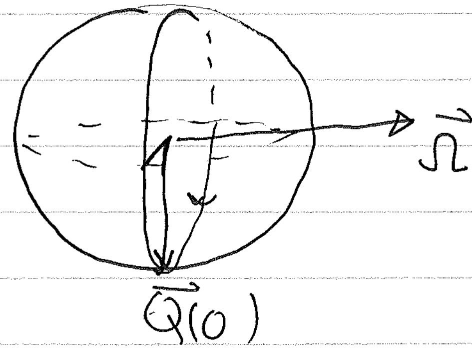
$$\vec{e}_n = \frac{\vec{\Omega}_{eff}}{|\vec{\Omega}_{eff}|} = \frac{\Delta}{\Omega_{eff}} \vec{e}_z + \frac{\Omega}{\Omega_{eff}} \vec{e}_\perp$$

<u>Consider case $\Delta = 0$ (on resonance)</u>

$$\tilde{H} = -\frac{\hbar\Omega}{2} \hat{\sigma}_\perp = -\frac{\hbar\Omega}{2} \left(\cos\phi\, \hat{\sigma}_x + \sin\phi\, \hat{\sigma}_y \right)$$



$\phi = 0$



$\phi = \frac{\pi}{2}$

Bloch vector rotates from spin-down to spin-up with frequency Ω ≡ Rabi flopping

On-resonance continued

General time-evolution $e^{-\frac{i}{\hbar}\tilde{H}t} = \hat{U}(t)$

$$\Rightarrow \hat{U}(t) = e^{i\frac{\Omega t}{2}(\sigma_x \cos\phi + \sigma_y \sin\phi)}$$

$$= \cos\frac{\Omega t}{2}\,\hat{\mathbb{1}} + i\sin\frac{\Omega t}{2}\left(\hat{\sigma}_x\cos\phi + \hat{\sigma}_y\sin\phi\right)$$

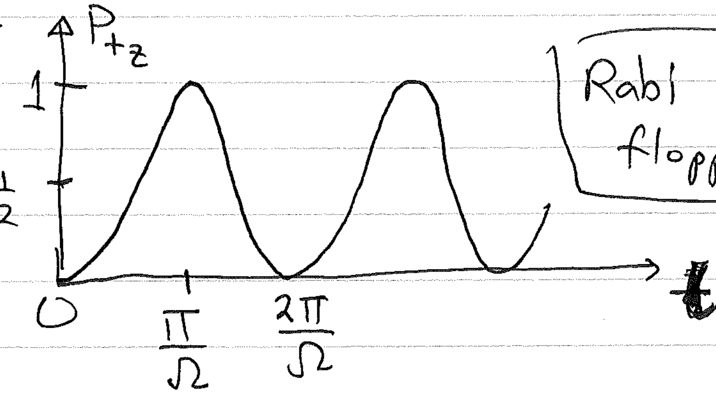
$$= \cos\frac{\Omega t}{2}\,\hat{\mathbb{1}} + i\sin\frac{\Omega t}{2}\left(\hat{\sigma}_+ e^{-i\phi} + \hat{\sigma}_- e^{i\phi}\right)$$

with $|\tilde{\psi}(0)\rangle = |-_z\rangle$

$$|\tilde{\psi}(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$$

$$= \cos\left(\frac{\Omega t}{2}\right)|-_z\rangle + i e^{-i\phi}\sin\left(\frac{\Omega t}{2}\right)|+_z\rangle$$

$$\boxed{P_{+z} = |\langle +_z|\tilde{\psi}(t)\rangle|^2 = \sin^2\left(\frac{\Omega t}{2}\right)}$$



Rabi flopping

$\boxed{\pi\text{-pulse}}$ $\Omega t = \pi \Rightarrow |\psi(t=\frac{\pi}{\Omega})\rangle = \underbrace{i e^{-i\phi}}_{\text{phase}}|+_z\rangle$

$\boxed{\frac{\pi}{2}\text{-pulse}}$ $\Omega t = \frac{\pi}{2} \Rightarrow |\psi(t=\frac{\pi/2}{\Omega})\rangle$

$$= \cos\frac{\pi}{4}|-_z\rangle + i\sin\frac{\pi}{4}e^{-i\phi}|+_z\rangle$$

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\left(|-_z\rangle + i e^{-i\phi}|+_z\rangle\right)$$

A $\frac{\pi}{2}$-pulse creates a 50-50 coherent superposition of $|\pm_z\rangle$ with a phase dependent on the ϕ of the oscillator

$\boxed{2\pi \text{ pulse}}$ $$|\psi(t=\tfrac{2\pi}{\Omega})\rangle = \cos\pi\,|-_z\rangle + i\sin\pi\, e^{-i\phi}\,|+_z\rangle$$

$$\Rightarrow \quad |\psi\rangle = -\,|-_z\rangle$$

-1 phase of SU(2)

<u>Finite detuning case</u>

$$\hat{\tilde{H}} = -\frac{\hbar\Omega_{eff}}{2}\,\hat{\sigma}_n$$

$$\hat{\sigma}_n = \vec{e}_n\cdot\hat{\vec{\sigma}} = \frac{\Delta}{\Omega_{eff}}\hat{\sigma}_z + \frac{\Omega}{\Omega_{eff}}\left(\hat{\sigma}_+ e^{-i\phi} + \hat{\sigma}_- e^{i\phi}\right)$$

$$\Rightarrow U(t) = e^{i\frac{\Omega_{eff}}{2}t\,\hat{\sigma}_n}$$

$$= \cos\frac{\Omega_{eff}}{2}t\,\hat{\mathbb{1}} - i\sin\frac{\Omega_{eff}}{2}t\,\hat{\sigma}_n$$

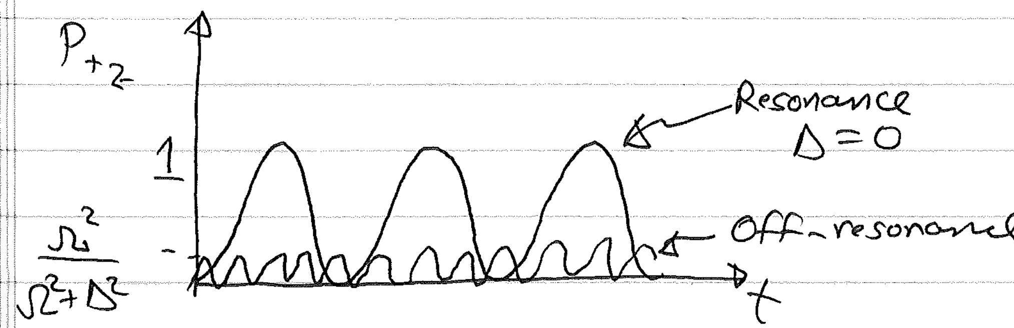
$$\boxed{\hat{U}(t) = \cos\frac{\Omega_{eff}}{2}t\,\hat{\mathbb{1}} - i\frac{\Delta}{\Omega_{eff}}\sin\frac{\Omega_{eff}}{2}t\,\hat{\sigma}_z - i\frac{\Omega}{\Omega_{eff}}\left(e^{-i\phi}\hat{\sigma}_+ + e^{i\phi}\hat{\sigma}_-\right)}$$

$\Rightarrow$ General solution

$$|\tilde{\psi}(t)\rangle = \hat{U}(t)\,|\tilde{\psi}(0)\rangle = \hat{U}(t)\,|-_z\rangle$$

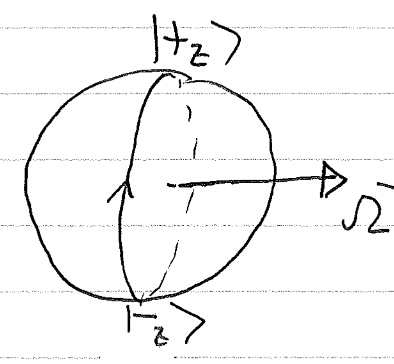
$$\Rightarrow \boxed{|\tilde{\psi}(t)\rangle = \left[\cos\left(\frac{\Omega_{eff}\,t}{2}\right) - i\,\frac{\Delta}{\Omega_{eff}}\sin\left(\frac{\Omega_{eff}\,t}{2}\right)\right]|-_z\rangle + \left[i e^{-i\phi}\,\frac{\Omega}{\Omega_{eff}}\sin\left(\frac{\Omega_{eff}\,t}{2}\right)\right]|+_z\rangle}$$

$$\boxed{P_{+z}(t) = |\langle +_z|\tilde{\psi}(t)\rangle|^2 = \frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2\left(\frac{\Omega_{eff}\,t}{2}\right)}$$



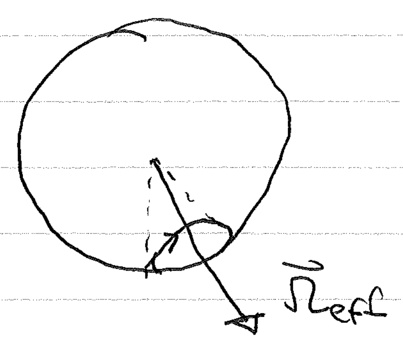
Rabi flopping!

On resonance



Full flop

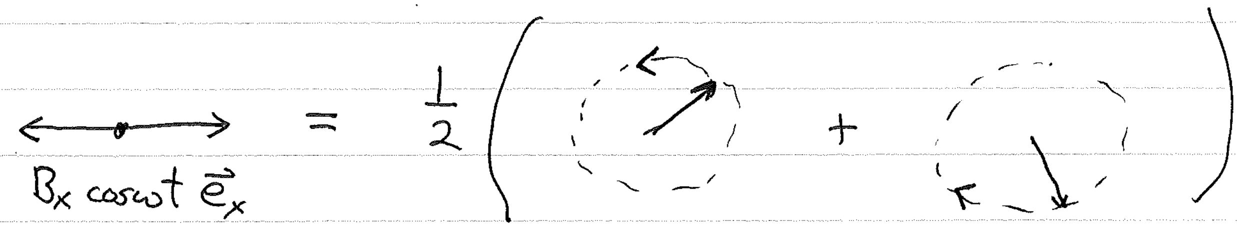
Off resonance



Partial flop

Rotating wave approximation (RWA)

Suppose that instead of a rotating transverse field we had a linearly oscillating field along x



$$= \underbrace{\frac{B_x}{2}\left(\cos\omega t\, \vec{e}_x + \sin\omega t\, \vec{e}_y\right)}_{\text{Co-rotating (resonant)}} + \underbrace{\frac{B_x}{2}\left(\cos\omega t\, \vec{e}_x - \sin\omega t\, \vec{e}_y\right)}_{\text{Counter-rotating (anti-resonant)}}$$

If $|\Delta| \ll \omega$ — near resonant and

$\Omega \ll \omega$ — slowly varying flip compared to rotation

then the counter-rotating terms rapidly oscillate on the scale of Rabi flopping

$\Rightarrow$ Counter-rotating terms average out and are negligible.

Under RWA

$$\hat{H}_{eff} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z - \frac{\hbar}{2}\gamma B_x(\hat{\sigma}_+ + \hat{\sigma}_-)\cos\omega t$$

$$\underset{\substack{\text{neglect}\\ \text{counter-rotations}}}{\approx} \frac{\hbar\omega_0}{2}\hat{\sigma}_z - \frac{\hbar}{2}\underbrace{\left(\frac{\gamma B_x}{2}\right)}_{=\,\Omega}\left(\hat{\sigma}_+ e^{-i\omega t} + \hat{\sigma}_- e^{i\omega t}\right)$$

Ω Rabi frequency with half of the amplitude