

## Lecture 4: The Optical Bloch equations (I)

Though <sup>for</sup> the problem of Rabi flopping, we could solve for the time evolution operator exactly, this is the exception rather than the rule. In most cases we must resort to approximate solutions to differential equations. To warm up to this and familiarize ourselves with some basic equation let's treat this for the Rabi flopping case.

### Three representations of state vector

• Probability amplitudes:  $|\Psi(t)\rangle = \tilde{c}_+(t) |+\rangle + \tilde{c}_-(t) |-\rangle$

$$\tilde{H} = -\frac{\hbar}{2} (\Delta \hat{\sigma}_z + \Omega \hat{\sigma}_x) \Rightarrow \frac{d}{dt} \begin{bmatrix} \tilde{c}_+ \\ \tilde{c}_- \end{bmatrix} = \frac{+i}{2} \begin{bmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{bmatrix} \begin{bmatrix} \tilde{c}_+ \\ \tilde{c}_- \end{bmatrix}$$

$$\dot{\tilde{c}}_+ = i \frac{\Delta}{2} \tilde{c}_+ + i \frac{\Omega}{2} \tilde{c}_-$$

$$\dot{\tilde{c}}_- = -i \frac{\Delta}{2} \tilde{c}_- + i \frac{\Omega}{2} \tilde{c}_+$$

(taking  $\phi=0$ )

$$\left[ \begin{array}{l} \text{On resonance } \dot{\tilde{c}}_+ = i \frac{\Omega}{2} \tilde{c}_- \quad \dot{\tilde{c}}_- = i \frac{\Omega}{2} \tilde{c}_+ \\ \Rightarrow \ddot{\tilde{c}}_+ = -\frac{\Omega^2}{4} \tilde{c}_+ \quad \text{SHO} \end{array} \right.$$

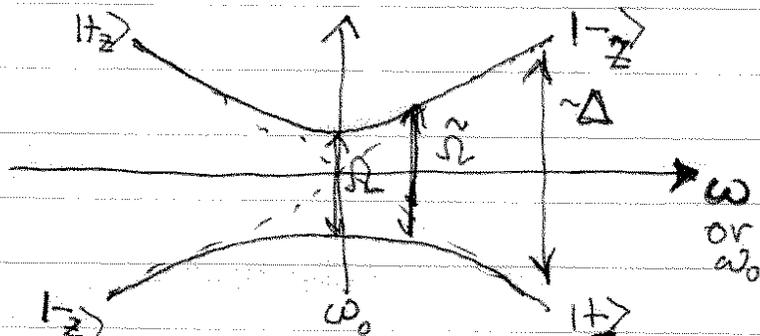
### Eigenspectrum

$$E = \pm \frac{\hbar}{2} \tilde{\Omega}, \quad \text{Eigenvectors}$$

$$| \pm_n \rangle = \cos \frac{\theta}{2} | \pm \rangle + i e^{i\phi} \sin \frac{\theta}{2} | \mp \rangle$$

"Dressed states"

$$\tan \theta = \frac{-\Omega}{-\Delta} \quad (\text{Fourth quadrant})$$



Anti-crossing

Resonant behavior

• Bloch vector  $\vec{Q} = \langle \vec{\sigma} \rangle = (\tilde{u}, \tilde{v}, \tilde{w})$

Heisenberg eq of motion  $\Rightarrow \frac{d\vec{Q}}{dt} = \vec{\Omega}_{\text{eff}} \times \vec{Q}$  "Gyroscope"

Rotation of Bloch vector about  $\vec{e}_n$

$$\Rightarrow \frac{d}{dt} \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} 0 & \Delta & 0 \\ -\Delta & 0 & \Omega \\ 0 & -\Omega & 0 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} \quad \begin{aligned} \dot{\tilde{u}} &= \Delta \tilde{v} \\ \dot{\tilde{v}} &= -\Delta \tilde{u} + \Omega \tilde{w} \\ \dot{\tilde{w}} &= -\Omega \tilde{v} \end{aligned}$$

"Bloch equations"

• Density matrix:  $\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}]$  "Master equation"

Note  $\tilde{\rho}_{++} + \tilde{\rho}_{--} = 1 \Rightarrow \dot{\tilde{\rho}}_{++} = -\dot{\tilde{\rho}}_{--} \quad \tilde{\rho}_{+-} = \tilde{\rho}_{+}^*$

$$\frac{d}{dt} \langle i | \hat{\rho} | j \rangle = -\frac{i}{\hbar} (\langle i | \hat{H} \hat{\rho} | j \rangle - \langle i | \hat{\rho} \hat{H} | j \rangle)$$

Another way:  $\langle i | \hat{\rho} | j \rangle = \tilde{c}_i \tilde{c}_j^*$  (for pure state)

$$\Rightarrow \frac{d}{dt} \tilde{\rho}_{+-} = \dot{\tilde{c}}_+ \tilde{c}_-^* + \tilde{c}_+ \dot{\tilde{c}}_-^* = i\Delta \tilde{c}_+ \tilde{c}_-^* + \frac{i}{2} \Omega (|\tilde{c}_+|^2 - |\tilde{c}_-|^2)$$

$$\Rightarrow \boxed{\dot{\tilde{\rho}}_{+-} = i\Delta \tilde{\rho}_{+-} + \frac{i}{2} \Omega (\tilde{\rho}_{++} - \tilde{\rho}_{--})}$$

$$\frac{d}{dt} \tilde{\rho}_{++} = \dot{\tilde{c}}_+ \tilde{c}_+^* + \tilde{c}_+ \dot{\tilde{c}}_+^* = i\frac{\Omega}{2} (\tilde{c}_- \tilde{c}_+^* - \tilde{c}_+ \tilde{c}_-^*)$$

$$\Rightarrow \dot{\tilde{\rho}}_{++} = i\frac{\Omega}{2} (\tilde{\rho}_{-+} - \tilde{\rho}_{+-})$$

$$\boxed{(\dot{\tilde{\rho}}_{++} - \dot{\tilde{\rho}}_{--}) = 2\dot{\tilde{\rho}}_{+-} = i\Omega (\tilde{\rho}_{-+} - \tilde{\rho}_{+-})}$$

Consistent?

$$\text{Recall } u = \langle \sigma_x \rangle = \rho_{+-} + \rho_{-+} = 2 \operatorname{Re} \rho_{+-}$$

$$v = \langle \sigma_y \rangle = \frac{\rho_{-+} - \rho_{+-}}{i} = -2 \operatorname{Im}(\rho_{+-})$$

$$\Rightarrow \rho_{+-} = \frac{1}{2}(u - iv) \quad \dot{\rho}_{+-} \stackrel{?}{=} \frac{1}{2}(\dot{u} - i\dot{v})$$

$$\dot{\rho}_{+-} = \frac{1}{2}(\Delta v + i\Delta u + i\Omega_B w)$$

$$\checkmark = i\Delta \left( \frac{u - iv}{2} \right) + i\frac{\Omega_B}{2} w = i\Delta \rho_{+-} + i\frac{\Omega_B}{2} (\rho_{++} - \rho_{--})$$

etc.

Atomic resonance as magnetic resonance

Optical spectroscopy (laser)

$$\hat{H} = \hat{H}_{\text{Atom}} + \hat{H}_{\text{AL}} + \cancel{(\hat{H}_{\text{AV}})} \rightarrow \text{neglect for now}$$

e.g. Hydrogen (alkali)

$$\hat{H}_{\text{Atom}} = \frac{\hat{p}_r^2}{2m_r} + V_{\text{coul}} + V_{\text{rel}} + \cancel{\hat{H}_{\text{cm}}} \rightarrow \text{neglect for now}$$

$$= \sum_j E_{njm_j} |n_j m_j\rangle \langle n_j m_j|$$

Interaction with the electromagnetic field

"Minimal coupling"  $\vec{p}_r \Rightarrow \vec{p}_r + \frac{e}{c} \vec{A}_L \leftarrow \text{laser}$

$$\hat{H} = \frac{\hat{p}_r^2}{2m_r} + \frac{e}{2mc} (\vec{p}_r \cdot \vec{A}_L + \vec{A}_L \cdot \vec{p}_r) + \frac{e^2}{2mc^2} \vec{A}_L^2$$

$$+ \hat{V}_{\text{coul}} + \hat{V}_{\text{rel}} = \hat{H}_A + \hat{H}_{\text{AL}}$$

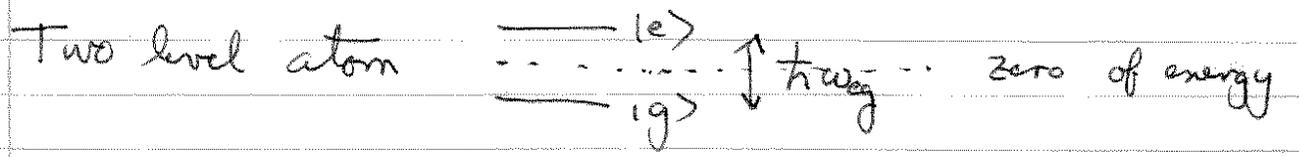
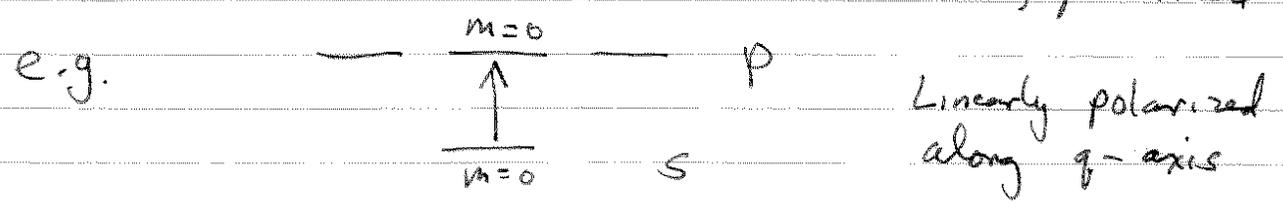
Multipole expansion  $\Rightarrow$  Electric dipole dominates for most transition

require  $\lambda \gg a_0$

$\Rightarrow$  For right gauge:  $\hat{H}_{AL} = -\hat{\mathbf{d}} \cdot \vec{E}_L(\vec{x}_A, t)$

$\hat{\mathbf{d}} = -e \hat{\mathbf{x}}_{rel}$  (electric dipole approx)

Two level approximation: Near resonant, polarized



$\Rightarrow \hat{H}_A = \frac{\hbar\omega_{eg}}{2} (|e\rangle\langle e| - |g\rangle\langle g|)$

$\hat{H}_{AL} = -\hat{\mathbf{d}} \cdot \vec{E}_L(\vec{x}_A, t)$

$\vec{E}_L = \vec{E}_L E_0 \cos(\phi(\vec{x}) - \omega_L t)$

$\phi(\vec{x}) = \vec{k}_L \cdot \vec{x} + \phi_0$

Monochromatic polarized plane wave

$\Rightarrow \hat{H}_{AL} = -(\hat{\mathbf{d}} \cdot \vec{E}_L) E_0 \cos(\phi(\vec{x}_A) - \omega_L t)$

Now  $\hat{\mathbf{d}} \cdot \vec{E}_L = \langle e | \hat{\mathbf{d}} \cdot \vec{E}_L | g \rangle |e\rangle\langle g| + \langle g | \hat{\mathbf{d}} \cdot \vec{E}_L | e \rangle |g\rangle\langle e|$

Only off-diagonal by parity  $\Rightarrow \hat{\mathbf{d}} \cdot \vec{E}_L = d_{eg} (|e\rangle\langle g| + |g\rangle\langle e|)$

where  $d_{eg} \equiv \langle e | \hat{d} \cdot \vec{E}_L | g \rangle$  real

$$\Rightarrow \begin{cases} \hat{H} = \hat{H}_A + \hat{H}_{AL} \\ \hat{H}_A = \hbar \omega_{eg} \frac{|e\rangle\langle e| - |g\rangle\langle g|}{2} \\ \hat{H}_{AL} = -d_{eg} E_0 (|e\rangle\langle g| + |g\rangle\langle e|) \cos(\omega_L t - \phi(\vec{x}_A)) \end{cases}$$

Two-level atom / field interaction  $\equiv$  Magnetic resonance

$$|g\rangle \equiv |-\rangle_z \quad |e\rangle \equiv |+\rangle_z$$

$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g| \quad \hat{\sigma}_+ = |e\rangle\langle g| \quad \hat{\sigma}_- = |g\rangle\langle e|$$

$$\Rightarrow \begin{cases} \hat{H}_A = \frac{\hbar \omega_{eg}}{2} \hat{\sigma}_z & \hat{H}_{AL} = -\frac{\hbar \Omega}{2} (\hat{\sigma}_+ + \hat{\sigma}_-) \cos(\phi(\vec{x}_A) - \omega_L t) \\ \text{where } \Omega = \frac{d_{eg} E}{\hbar} = \text{Rabi freq} \end{cases}$$

Make the RWA  $\Rightarrow$  Neglect anti-resonant terms

$$\hat{H} = \frac{\hbar \omega_{eg}}{2} \hat{\sigma}_z - \frac{\hbar \Omega}{2} (\hat{\sigma}_+ e^{i\phi(\vec{x}_A)} e^{-i\omega_L t} + \hat{\sigma}_- e^{-i\phi(\vec{x}_A)} e^{i\omega_L t})$$

Go to rotating frame

$$\hat{H} \approx -\frac{\hbar \Delta}{2} \hat{\sigma}_z - \frac{\hbar \Omega}{2} (\hat{\sigma}_+ e^{i\phi(\vec{x}_A)} e^{-i\omega_L t} + \text{c.c.})$$

$$\hat{H}_{\text{eff}} = \frac{\hbar \vec{\Omega}_{\text{eff}}}{2} \cdot \hat{\sigma} \quad \vec{\Omega}_{\text{eff}} = -(\Delta \hat{e}_z + \Omega (\cos\phi \hat{e}_x + \sin\phi \hat{e}_y))$$

Suppose at  $t=0$ , atom is in ground state (take  $\phi=0$ )

$$|\tilde{\Psi}(t)\rangle = \left[ \cos\left(\frac{\tilde{\Omega}t}{2}\right) - i \frac{\Delta}{\tilde{\Omega}} \sin\left(\frac{\tilde{\Omega}t}{2}\right) \right] |g\rangle + \left[ i \frac{\Omega}{\tilde{\Omega}} \sin\left(\frac{\tilde{\Omega}t}{2}\right) \right] |e\rangle$$

Rabi flopping

In "lab frame"

$$\hat{d}(t) = D^\dagger \hat{d} D = \text{deg} \left[ e^{+i\omega_L t} |e\rangle\langle g| + e^{-i\omega_L t} |g\rangle\langle e| \right]$$

$$\Rightarrow \langle \hat{d}(t) \rangle = \text{deg} \ 2\text{Re} \left( e^{i\omega_L t} \tilde{c}_g \tilde{c}_e^* \right)$$

$$= \text{deg} \ 2\text{Re} \left( e^{i\omega_L t} \tilde{\rho}_{ge} \right)$$

$$= \text{deg} \ \text{Re} \left( e^{i\omega_L t} (\tilde{u} + i\tilde{v}) \right)$$

$$= \text{deg} \left( \tilde{u}(t) \cos\omega_L t - \tilde{v}(t) \sin\omega_L t \right)$$

↑  
"in phase"

↑  
"in quadrature"

$$\tilde{u}(t) = 2 \text{Re}(\tilde{c}_g \tilde{c}_e^*) = \frac{\Omega \Delta}{\tilde{\Omega}} (1 - \cos(\tilde{\Omega}t))$$

$$\tilde{v}(t) = -\frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega}t)$$

On resonance:  $\tilde{u} = 0$        $\tilde{v} = -\sin(\tilde{\Omega}t)$       max possible

In phase component  $\Rightarrow$  real index of fraction  
"reactive part"

In quadrature  $\Rightarrow$  imaginary part of index  
"absorption"

Radiation Spectrum    AM modulation at  $\tilde{\Omega} \Rightarrow$  Rabi sidebands  
Mollow triplet