

Lecture 12 Optical Coherence and Statistical Optics

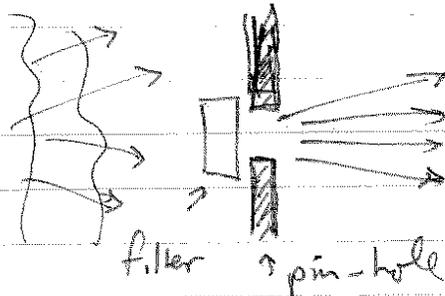
Classical statistical optics (Born and Wolf)

"Natural light sources" (e.g. stars) radiate fields whose amplitude and phase fluctuate:

Statistical Uncertainty

Generally we consider quasimonochromatic / paraxial

$$E^{(+)}(\vec{x}, t) = \mathcal{E}(\vec{x}, t) e^{i\vec{k}_0 \cdot \vec{x} - \omega_0 t}$$



$$\frac{|\nabla \mathcal{E}|}{\mathcal{E}} \ll k_0$$

$$\frac{|\dot{\mathcal{E}}|}{\mathcal{E}} \ll \omega_0$$

Fundamental object $P(\{\mathcal{E}\}, t)$: Probability distribution over field configurations ~~or~~ Fourier modes at time t

Measurements of function $f(\mathcal{E})$

Expected value $\langle f(\mathcal{E}) \rangle_t = \int d\{\mathcal{E}\} P(\{\mathcal{E}\}, t) f(\mathcal{E})$

Stationarity: Statistics do not change with time $P(\{\mathcal{E}\})$

Ergodicity: Average over time \equiv Expectation

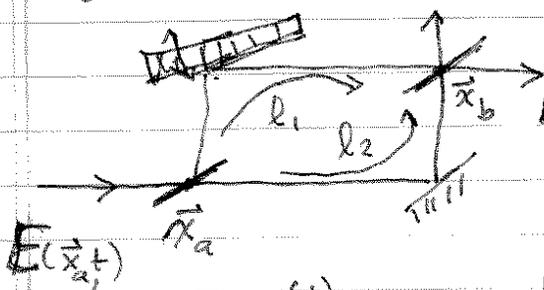
$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt f(\mathcal{E}(\vec{x}, t)) = \langle f \rangle$$

Coherence: Fields have well defined phase relation

Temporal - Variation in time at same position

Spatial - Variation in space at same time

Eg. Mach-Zehnder measures temporal coherence (auto-correlation)



$E(\vec{x}_b, t) =$ Superposition of two fields at two times

$$E^{(+)}(\vec{x}_b, t) = \left[E^{(+)}\left(\vec{x}_a, \underbrace{t - \frac{l_1}{c}}_{t_1}\right) + E^{(+)}\left(\vec{x}_a, \underbrace{t - \frac{l_2}{c}}_{t_2}\right) \right]$$

Output Intensity (Ergodic assumption)

$$I = \langle |E^{(+)}(\vec{x}_b, t)|^2 \rangle = \langle E^{(-)}(\vec{x}_b, t) E^{(+)}(\vec{x}_b, t) \rangle$$

$$= \left(G^{(1)}(\vec{x}_a t_1; \vec{x}_a t_1) + G^{(1)}(\vec{x}_a t_2; \vec{x}_a t_2) \right.$$

$$\left. + G^{(1)}(\vec{x}_a t_1; \vec{x}_a t_2) + G^{(1)}(\vec{x}_a t_2; \vec{x}_a t_1) \right) / 4$$

where $G^{(1)}(\vec{x}t; \vec{x}'t') \equiv \langle E^{(-)}(\vec{x}t) E^{(+)}(\vec{x}'t') \rangle$

"First order correlation"

$$= G^{(1)}(\vec{x}'t'; \vec{x}t)^*$$

Consider: $E^{(+)}(\vec{x}, t) = \mathcal{E}(t) e^{ik_0 x - \omega_0 t}$

$$G^{(1)}(\vec{x}t; \vec{x}t') = \underbrace{\langle \mathcal{E}^*(t) \mathcal{E}(t') \rangle}_{\Gamma(t, t')} e^{i\omega_0(t-t')} \quad \text{Stationary}$$

Note: $G^{(1)}(\vec{x}t, \vec{x}t) = \langle |\mathcal{E}(t)|^2 \rangle = I_0$ (input intensity)

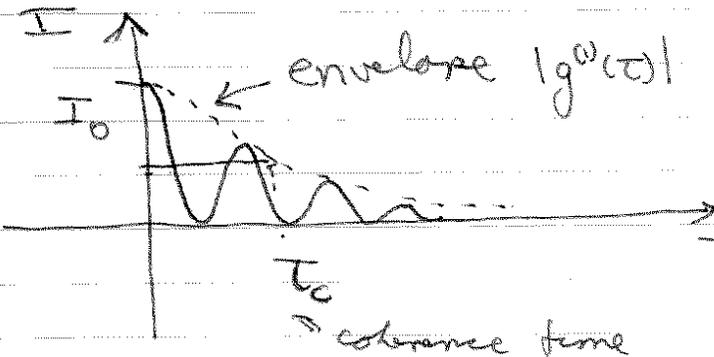
Define normalized correlation function

$$g(\vec{x}_1, t_1; \vec{x}_2, t_2) = \frac{G^{(1)}(\vec{x}_1, t_1; \vec{x}_2, t_2)}{\sqrt{G^{(1)}(\vec{x}_1, t_1) G^{(1)}(\vec{x}_2, t_2)}}$$

$$\Rightarrow I = I_0 \left(\frac{1 + \text{Re}(g^{(1)}(\tau))}{2} \right)$$

where $g^{(1)}(\tau) = \frac{\langle \mathcal{E}^*(\tau) \mathcal{E}(0) \rangle}{I_0} e^{i\omega_0 \tau} = |g^{(1)}(\tau)| e^{i(\omega_0 \tau + \phi(\tau))}$

$$\Rightarrow I = I_0 \left(\frac{1 + |g^{(1)}(\tau)| \cos(\omega_0 \tau + \phi(\tau))}{2} \right)$$



$$\tau = \frac{l_2 - l_1}{c}$$

$l_c = c\tau_c =$ coherence length

Fringe visibility

$$V(\tau) = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{I_0(1+|g^{(1)}(\tau)|) - I_0(1-|g^{(1)}(\tau)|)}{[\quad] + [\quad]}$$

$$\Rightarrow \boxed{V(\tau) = |g^{(1)}(\tau)|} \quad \begin{array}{l} \text{Measuring visibility} \\ \Rightarrow \text{Measuring correlation} \end{array}$$

Wiener-Khinchin theorem

Consider Fourier transform of correlation

$$\int_{-\infty}^{\infty} d\tau \underbrace{G^{(1)}(\tau)}_{\parallel} e^{i\omega\tau} = \langle |\tilde{E}(\omega)|^2 \rangle$$

↑ spectral energy density

$$\langle \underbrace{E^*(\tau)}_{\parallel} E(0) \rangle$$

$$\int dt' E^*(\tau+t') E(t') \leftarrow \text{convolution}$$

$\Rightarrow g^{(1)}(\tau) =$ Fourier transform of normalized spectral density function

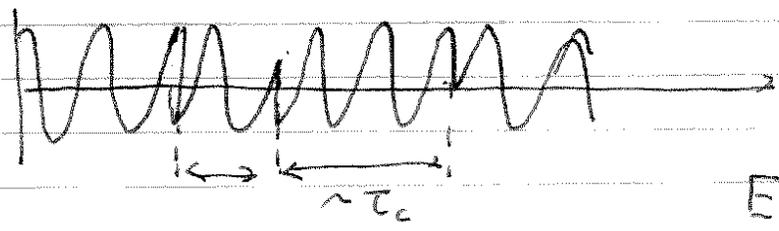
Fourier duality $\tau_c \sim \frac{1}{\Delta\omega}$ (bandwidth)

$$\Rightarrow l_c \sim \frac{c}{\Delta\omega}$$

Statistics of "natural" light source

Examples: Stars, discharge lamp, light bulb

"Collision broadened" source



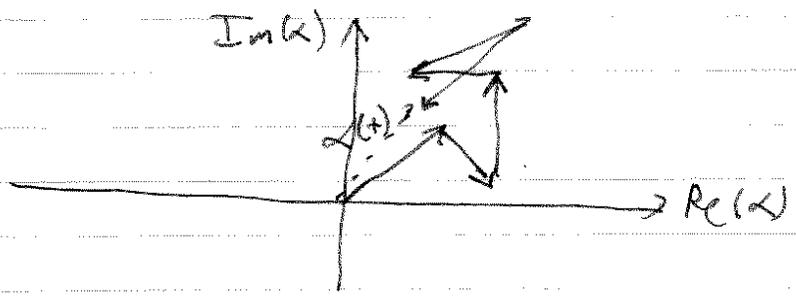
Wave train of single atom

$$E(t) = E_0 e^{-i\omega t} e^{i\phi(t)}$$

↑
random

$$\Rightarrow E(t) = \sum_{i=1}^N E_i(t) \quad (N \text{ atoms} \approx \text{Avagadro's } \#)$$

$$= (E_0 e^{-i\omega_0 t}) (\alpha(t)) \quad \alpha(t) = \sum_{i=1}^N e^{i\phi_i(t)}$$



random walk

Random walk \Rightarrow "Wiener Process"

$$\Rightarrow P(\alpha(t)) = \frac{1}{\sqrt{\pi N}} e^{-|\alpha(t)|^2 / N}$$

Central limit theorem

"Gaussian random process"

$$\Rightarrow P(E) = \text{Gaussian function of } |E|$$

Note: Spectrum need not be Gaussian. Do not confuse distribution of $|E|$ with distribution of Fourier components

General properties of the coherence function

$$|G^{(1)}(x_1; x_2)|^2 \leq G^{(1)}(x_1; x_1) G^{(1)}(x_2; x_2)$$

Proof $G^{(1)}(x_1; x_2) = \int d\epsilon \epsilon^3 P(\epsilon) E^*(x_1) E(x_2)$
 $= (E(x_1) | E(x_2))$ Inner product

Cauchy-Schwartz inequality:

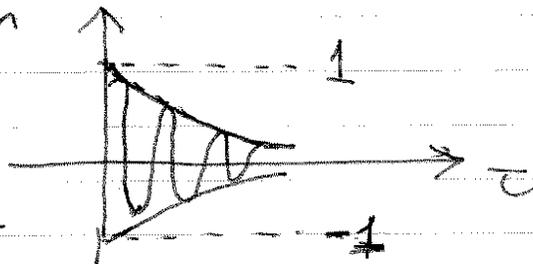
$$|(E(x_1) | E(x_2))|^2 \leq \|E(x_1)\|^2 \|E(x_2)\|^2$$

$G^{(1)}(x_1; x_1) \quad G^{(1)}(x_2; x_2) \quad \text{g.e.d.}$

Thus $|g^{(1)}(x_1, x_2)| \leq 1$

e.g. Lorentian spectrum

\Rightarrow Exponentially
decaying
auto-correlation



Ideal coherent source $\Rightarrow \infty$ coherence length

\Rightarrow (First order) Coherence $|g^{(1)}(\omega)| = 1$

Question: What characterizes those ~~states~~ states for which the field exhibits coherence?

Coherence and Factorizability

Suppose $G^{(1)}(x_1, x_2) = E^*(x_1) E(x_2)$

$$\begin{aligned} \Rightarrow |g^{(1)}(x_1, x_2)| &= \frac{|G^{(1)}(x_1, x_2)|}{\sqrt{G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2)}} = \\ &= \frac{|E(x_1)| |E(x_2)|}{\sqrt{|E(x_1)|^2 |E(x_2)|^2}} = \underline{1} \end{aligned}$$

Factorizability \Rightarrow Coherence

In fact it goes the other way

Coherence \Rightarrow Factorizability

$$g^{(1)}(x_1, x_2) = 1 \Rightarrow G^{(1)}(x_1, x_2) = E^*(x_1) E(x_2)$$

(see Glauber lect notes)

\therefore Coherence \Leftrightarrow Factorization of correlation function

Note: For stationary fields that are first order coherent

$$G^{(1)}(t_1, t_2) = G^{(1)}(t_1 - t_2) = E^*(t_1) E(t_2)$$

$$\Rightarrow E(t) \sim e^{-i\omega t} \Rightarrow \text{Monochromatic}$$

First order coherence \Rightarrow monochromatic

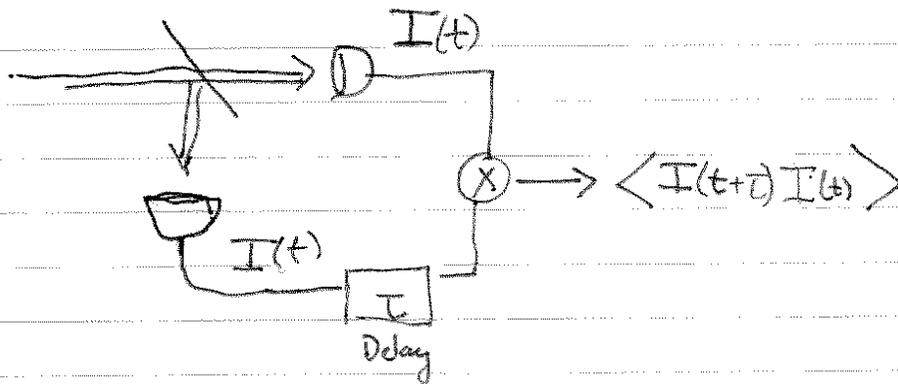
Intensity-Intensity Correlations:
Hanbury-Brown and Twiss Experiment

Suppose our interferometer has very large path lengths. Generally, it is difficult to keep the interferometer balanced since a small fluctuation in path difference Δl leads to shift in the rapidly oscillating interference pattern.

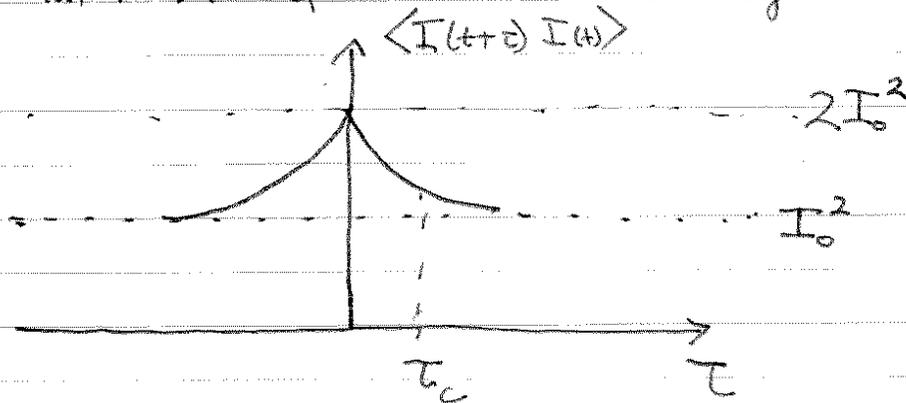
In ~~the~~ 1956-7 R. Hanbury Brown and R.Q. Twiss
(Nature 177 27 (1956); Proc. Roy. Soc. (London) A242 300 (1957))
ibid 291 (1957)

Employed a new technique to measure coherence.

Consider "correlator"



The data showed (for a discharge lamp)



The intensity auto-correlation showed the coherence envelope! Technically this was much easier for very long armed interferometer since $I(t)$ varies at much slower frequencies than the field and is thus much less susceptible to noise. Historically HBT first did their experiment to measure spatial coherence of stars (see Glauber Notes)

How can this be? How can we get phase information (relating to the coherence envelope) by beating together photon currents! Does this violate Dirac's famous analysis of a Michelson Interferometer where he says "each photon only interferes with itself; interference between different photons never occurs."

Historically the HBT caused great controversy. In 1963 R. Glauber wrote his seminal paper *Phy Rev.* 131 (2766) explaining HBT using the quantum theory of ~~measure~~ photon counting. We will explore this next lecture.

Glauber showed that HBT could be understood classically

$$\begin{aligned} \text{Consider } \langle I(t+\tau) I(t) \rangle &= \\ &= \langle |E^*(t+\tau)|^2 |E(t)|^2 \rangle \\ &= \langle E^*(t+\tau) E^*(t) E(t+\tau) E(t) \rangle \end{aligned}$$

(12.10)

The HBT signal is a second order correlation

$$G^{(2)}(x_1, x_2; x_1', x_2') \equiv \langle E^*(x_1) E^*(x_2) E(x_1) E(x_2) \rangle$$

$$g^{(2)}(x_1, x_2; x_1', x_2') \equiv \frac{G^{(2)}(x_1, x_2; x_1', x_2')}{\sqrt{G^{(1)}(x_1, x_1') G^{(1)}(x_1, x_2') G^{(1)}(x_2, x_1') G^{(1)}(x_2, x_2')}}}$$

Now for Gaussian statistics all correlation functions are ~~not~~ determined by the lowest order moments

$$G^{(2)}(x_1, x_2; x_1', x_2') = G^{(1)}(x_1, x_1') G^{(1)}(x_2, x_2') + G^{(1)}(x_1, x_2') G^{(1)}(x_2, x_1') \\ \text{(sum over all pairs)}$$

(Proof: Use characteristic function, see Glauber)

thus

$$\langle E^*(t+\tau) E^*(t) E(t+\tau) E(t) \rangle$$

$$= \langle |E(t+\tau)|^2 \rangle \langle |E(t)|^2 \rangle + \langle E^*(t+\tau) E(t) \rangle \langle E^*(t) E(t+\tau) \rangle$$

stationary

$$= \underbrace{\langle |E(t)|^2 \rangle^2}_{I_0} + \underbrace{\langle E^*(\tau) E(0) \rangle}_{G^{(1)}(\tau)} \underbrace{\langle E^*(0) E(\tau) \rangle}_{G^{(1)}(\tau)^*}$$

$$= \boxed{I_0^2 (1 + |g^{(1)}(\tau)|^2)} \quad \text{HBT signal!}$$

The coherence envelope is thus measured as the statistical fluctuations in the intensity,

$$\langle \delta I(\tau) \delta I(0) \rangle = \langle (I(\tau) - \langle I(\tau) \rangle) (I(0) - \langle I(0) \rangle) \rangle$$

$$= \underbrace{\langle I(\tau) I(0) \rangle}_{\text{correlation}} - \underbrace{\langle I(\tau) \rangle \langle I(0) \rangle}_{\text{statistically independent}}$$

$$= I_0^2 (1 + |g^{(2)}(\tau)|^2) - I_0^2$$

$$\Rightarrow |g^{(2)}(\tau)|^2 = \frac{\langle \delta I(\tau) \delta I(0) \rangle}{\langle I \rangle^2}$$

In the next lecture we will explore the quantum theory to see how it contrast with these classical predictions.