Lecture 24: Quantum Trajectories - The Quantum Jump Picture

In our studies we have seen the need for density operators describing mixed states in two different circumstances

(i) State prepared "statistically" with classical probabilities

$$\hat{\rho} = \sum_i P_i |\psi_i\rangle\langle\psi_i|$$

(ii) State is a "subsystem" of a larger set of degrees of freedom that are entangled

$$\hat{\rho}_S = \mathrm{Tr}_R \left(|\Psi_{SR}\rangle\langle\Psi_{SR}| \right)$$

e.g. system + reservoir

Are these two pictures related? In a certain sense, yes. We can think of the reservoir (or environment) as performing a measurement of system but not telling us the result. This is equivalent to "Alice" preparing a state for "Bob" but not telling him the result, only the probabilities he should expect. The interpretation, of course, is steeped in the deep issues of quantum measurement theory, but these ~~at~~ need not concern ~~us~~ now.

Measurements of a quantum system cause "back-action" or "collapse" of the state, with associated randomness. The continuously measured quantum state will thus evolve stochastically. This evolution has come to be known as a "quantum trajectory", a term originally coined by H.J. Carmichael.

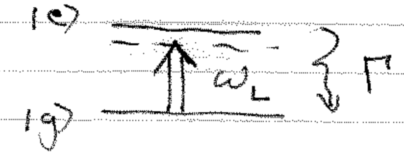
The density operator is then the average over many different quantum trajectories, a particular set known as an unravelling.

The development of stochastic wave function evolution proceeded in parallel along a number of fronts in the late 1980's and 1990's and falls under many names including "quantum jumps", "Quantum Monte Carlo Wavefunctions", "quantum trajectories" and "quantum state diffusion". Today we understand the connections between all of these in modern measurement theory to be discussed soon.

We begin with the quantum jump picture

<u>Spontaneous emission and jumps</u>

Consider our favorite problem of the driven 2-lev atom



The master equation for the atom, having traced over the vacuum in the Born-Markov approx

$$\frac{d\hat{\rho}}{dt} = \underbrace{-\frac{i}{\hbar}[\hat{H},\hat{\rho}]}_{\text{Hamiltonian}} \underbrace{- \frac{\Gamma}{2}(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-)}_{\text{Decay}} + \underbrace{\Gamma\hat{\sigma}_-\rho\hat{\sigma}_+}_{\text{Feeding}}$$

$$\hat{H} = -\hbar\frac{\Delta}{2}\underbrace{\hat{\sigma}_+\hat{\sigma}_-}_{|e\rangle\langle e|} - \hbar\frac{\Omega}{2}(\hat{\sigma}_+ + \hat{\sigma}_-) \quad \text{in RWA}$$

Note that the Hamiltonian and decay terms can be lumped together to form an effective <u>non-Hermitian</u> Hamiltonian

$$\hat{H}_{eff} \equiv \hat{H} - i\frac{\hbar\Gamma}{2}\hat{\sigma}_+\hat{\sigma}_- = \hbar\Big(-\frac{\Delta}{2} - i\underbrace{\frac{\Gamma}{2}}_{\substack{\Uparrow \\ \text{imaginary energy} \\ \Rightarrow \text{Like term}}}\Big)\hat{\sigma}_+\hat{\sigma}_- + \hat{H}_{AL}$$

the $\left.\frac{d\hat{\rho}}{dt}\right|_{Ham} + \left.\frac{d\hat{\rho}}{dt}\right|_{decay} \equiv \frac{-i}{\hbar}[\hat{H}_{eff}, \hat{\rho}]^{\prime}$

where $^{\prime}$ denote $\equiv \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{+}\right)$

$$= -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{\Gamma}{2}\{\hat{\sigma}_{+}\hat{\sigma}_{-}, \hat{\rho}\} \Leftarrow \text{anti-commutator}$$

$$\Rightarrow \boxed{\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_{eff}, \hat{\rho}]^{\prime} + \Gamma\hat{\sigma}_{-}\hat{\rho}\hat{\sigma}_{+}}$$

Now suppose $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$, a pure state

$$\hat{\rho}(t+dt) = \hat{\rho}(t) - \frac{i}{\hbar}[\hat{H}_{eff}, \hat{\rho}]^{\prime}dt + \Gamma dt\, \hat{\sigma}_{-}\hat{\rho}\hat{\sigma}_{+}$$

$$\simeq \left(1 - \frac{i}{\hbar}\hat{H}_{eff}\,dt\right)|\psi(t)\rangle\langle\psi(t)|\left(1 + \frac{i}{\hbar}\hat{H}_{eff}^{+}\,dt\right)$$

$$+ \left(\sqrt{\Gamma dt}\,\hat{\sigma}_{-}\right)|\psi(t)\rangle\langle\psi(t)|\left(\sqrt{\Gamma dt}\,\hat{\sigma}_{+}\right)$$

At this later time $\hat{\rho}$ is a statistical mixture

$$\hat{\rho}(t+dt) = (1-dp(t))|\phi_0\rangle\langle\phi_0| + dp(t)|\phi_1\rangle\langle\phi_1|$$

where $dp(t) = \Gamma dt\, \langle\psi(t)|\hat{\sigma}_{+}\hat{\sigma}_{-}|\psi(t)\rangle$

$$= \Gamma dt\, |\langle e|\psi(t)\rangle|^2 = P_e(t)\,\Gamma dt$$

$$|\phi_1(t)\rangle = \frac{\sqrt{\Gamma dt}\,\hat{\sigma}_{-}|\psi(t)\rangle}{\sqrt{dp(t)}} = \frac{\hat{\sigma}_{-}|\psi(t)\rangle}{\|\hat{\sigma}_{-}|\psi(t)\rangle\|}$$

$$|\phi_0(t)\rangle = \left(1 - \frac{i}{\hbar}\hat{H}_{eff}\,dt\right)|\psi(t)\rangle \Big/ \sqrt{1 - dp(t)}$$

Proof: Define $|\tilde{\phi}_1\rangle \equiv \sqrt{\Gamma dt}\ \hat{\sigma}_-|\psi(t)\rangle$

$$|\tilde{\phi}_0\rangle \equiv \left(1-\frac{i}{\hbar}\hat{H}_{eff}\,dt\right)|\psi(t)\rangle$$

$$\Rightarrow \hat{\rho}(t+dt) = |\tilde{\phi}_0\rangle\langle\tilde{\phi}_0| + |\tilde{\phi}_1\rangle\langle\tilde{\phi}_1| \quad \left(\text{to first order in } dt\right)$$

$$= \|\tilde{\phi}_0\|^2\ |\phi_0(t)\rangle\langle\phi_0(t)| + \|\tilde{\phi}_1\|^2\ |\phi_1(t)\rangle\langle\phi(t)|$$

$$\|\tilde{\phi}_1\|^2 = \langle\tilde{\phi}_1|\tilde{\phi}_1\rangle = \Gamma dt\ \langle\psi(t)|\hat{\sigma}_+\hat{\sigma}_-|\psi(t)\rangle = dp(t)\ \checkmark$$

$$\|\tilde{\phi}_0\|^2 = \langle\tilde{\phi}_0|\tilde{\phi}_0\rangle = 1 - \frac{i}{\hbar}\langle\psi(t)|\left(\hat{H}_{eff}-\hat{H}_{eff}^{\dagger}\right)|\psi(t)\rangle dt$$

$$= 1 - \Gamma dt\,\langle\psi(t)|\hat{\sigma}_+\hat{\sigma}_-|\psi(t)\rangle = 1 - dp(t)\ \checkmark$$

We can interpret this result as follows

- With probability $dp(t) = P_e(t)\,\Gamma dt$ the system undergoes a "<u>quantum jump</u>"

$$|\psi(t)\rangle \Rightarrow \frac{\hat{\sigma}_-|\psi(t)\rangle}{\|\hat{\sigma}_-|\psi(t)\rangle\|} = |g\rangle\langle e|\frac{\left(c_e(t)|e\rangle + c_g(t)|g\rangle\right)}{|c_e(t)|}$$

$$= |g\rangle$$

The system is "prepared in the ground state" by the emission of a photon

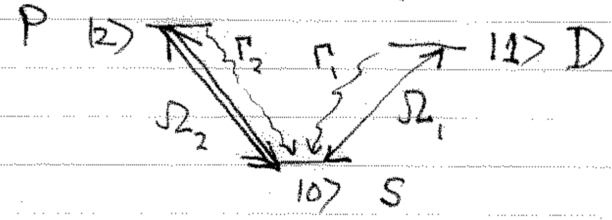
- With probability $(1-dp(t))$ <u>no jump occurs</u> Under that circumstance the state evolves continuously according to the effective Hamiltonian

$$|\psi(t)\rangle \Rightarrow \frac{e^{-i\frac{H_{eff}}{\hbar}dt}\ |\psi(t)\rangle}{\|\tilde{\phi}_0(t)\|}$$

Normalization is required for NonHermitian evolution.

The notion of quantum jumps has played an important role in the developement of the notion quantum trajectories

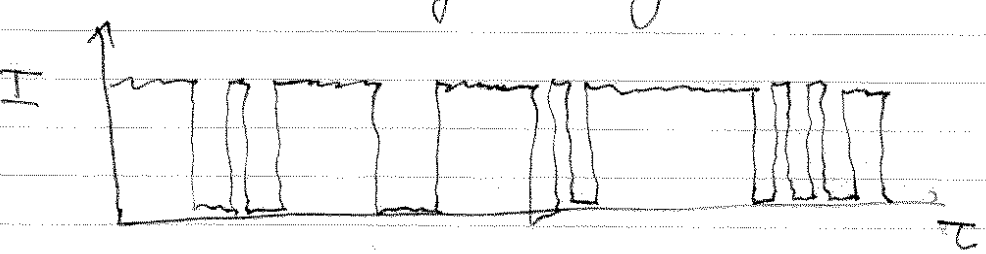
Consider a "V-system"



- $|0\rangle - |2\rangle$ is a strong dipole allowed transition with rapid decay rate Γ_2
- $|0\rangle - |1\rangle$ is a weak quadrapole transition with very slow (meta-stable) decay Γ_1

In 1975 Dehmelt proposed such a system to measure the ~~[illegible]~~ very narrow line width Γ_1. Termed "electron shelving", the strong transition "amplifies" the weak transition. Everytime state ~~[illegible]~~ $|1\rangle$ decays (say every second), 10^9 photons/sec are scattered on the $|0\rangle \to |2\rangle$ transition.

In 1985 Kimble and Cook showed how the fluorescence signal at ω_{02} ~~would exhibit~~ for a single atom would exhibit a "random telegraph" signal with the atom randomly blinking on and off



This was observed by Wineland's group ~~has~~ in 1986 for a single traped Hg atom

Every time the signal is turned on, it is triggered by a "quantum jump" from $|0\rangle \to |1\rangle$. Zoller's analysis in 1987 was one path to quantum trajectories.

Evolution under the no jump condition

Under the condition that no jump occurs, the wave function evolves under the action of a non-Hermitian Hamiltonian. How do we interpret this?

Consider the simplest case of the atom coupled to the vacuum in the absense of a driving laser field.

Suppose at time t $|\psi(t)\rangle = c_g|g\rangle + c_e|e\rangle$

$$\Rightarrow |\tilde{\psi}(t+\delta t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_{eff}\delta t}|\tilde{\psi}(t+\delta t)\rangle$$

$$= c_g|g\rangle + e^{-i\omega_{eg}\delta t - \frac{\Gamma}{2}\delta t} c_e|e\rangle$$

$$\Rightarrow |\psi(t+\delta t)\rangle = \frac{|\tilde{\psi}(t+\delta t)\rangle}{\|\tilde{\psi}(t+\delta t)\|} = c_g\left(1+\frac{\Gamma}{2}\delta t|c_e|^2\right)|g\rangle$$

$$+ e^{-i\omega_{eg}\delta t}\left(1-\frac{\Gamma\delta t}{2}|c_g|^2\right)c_e|e\rangle$$

(to first order in $\Gamma\delta t$)

Thus we see that in addition to the phase evolution there is a small rotation of the state from $|e\rangle$ to $|g\rangle$. This has the following interpretation.

No jump corresponds to no detection of a photon in the vacuum. This null detection gives us some information. With this information we must readjust our quantum state (which represents our knowledge of the atom). Not seeing a photon, we are more likely to have a ground state atom than an excited state atom, hence the decay of the excited state probability and growth of the ground states amplitude

The rotation of the state associated with a null detection is essential to get proper statistics.

If it did not occur than the probability for observing a jump would be constant with time

$$dp(t) = P_e(t)\,\Gamma\,dt = |c_e|^2\,\Gamma\,dt = \text{constant}$$

Thus if $|\psi(0)\rangle = c_g|g\rangle + c_e|e\rangle$ we would always see a jump at some point, whereas we know from ~~jump~~ quantum mechanics that given this initial state we expect no jump from $t=0 \to \infty$ with probability $|c_g|^2$. The rotation for null measurement ensures this.

If no jump occurs from $0 \to t$ we have

$$|\psi(t)\rangle = \frac{c_g|g\rangle + e^{-i\omega_0 t}\, c_e\, e^{-\Gamma t/2}\,|e\rangle}{\sqrt{|c_g|^2 + |c_e|^2 e^{-\Gamma t}}}$$

$\Rightarrow$ Probability for no jump between 0 and t $P(t)$ satisfies

$$P(t+\delta t) = \left(P_{\text{no jump } t \to t+\delta t}\right) \cdot P_{\text{no jump } 0 \to t}$$

$$= \left(1 - |\langle e|\psi(t)\rangle|^2\,\Gamma\,\delta t\right) P(t)$$

$$= \left(1 - \frac{\Gamma\,\delta t\,|c_e|^2\,e^{-\Gamma t}}{(|c_g|^2 + |c_e|^2 e^{-\Gamma t})}\right) P(t)$$

$$\Rightarrow \boxed{P(t) = |c_g|^2 + e^{-\Gamma t}\,|c_e|^2}$$

As $t \to \infty$ $P \to |c_g|^2$ as expected