

## Lecture 27: Quantum Trajectories and Formal Measurement Theory

We saw in the last lecture how the Master equation could be unravelled in different ways depending upon what "gedanken measurement" was performed on the environment. In particular we show that if the Lindblad relaxation operator was invariant under some symmetry

$$\hat{T} L_{\text{relax}}[\hat{\rho}] \hat{T}^\dagger = L_{\text{relax}}[\hat{T} \hat{\rho} \hat{T}^\dagger],$$

then the two sets of jump operators

$$\{\hat{L}_j\} \quad \text{and} \quad \{\hat{Q}_j = \hat{T}^\dagger \hat{L}_j \hat{T}\} \quad \text{lead to}$$

different unravellings, but the same master equation.

We would like to know, in general, which different unravellings yield the same density operator. To

do so we must delve into the formal theory

of quantum measurement. Most of us are familiar

with this in its simplest form, the Von Neumann

projection postulate. It turns out this is

not the whole story.

Ensemble Decomposition

We have seen, since the beginning of the course, that a preparation of a pure state amongst a set "ensemble"  $\{|\psi_i\rangle\}$  with probabilities  $\{p_i\}$  yields a mixed state described by the density operator

$$\hat{\rho} = \sum_{i=1}^N p_i |\psi_i\rangle \langle \psi_i|$$

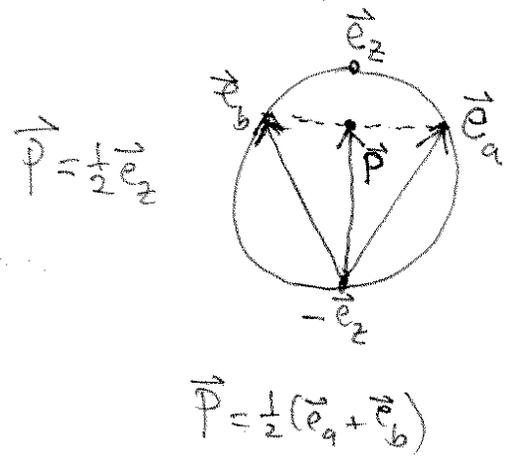
This representation of  $\hat{\rho}$  is not unique. That is many different different ensemble preparations  $\{q_i, |\phi_i\rangle\}$  yield the same density matrix

$$\hat{\rho} = \sum_{j=1}^M q_j |\phi_j\rangle \langle \phi_j| \quad M \text{ need not equal } N$$

We have seen this in the simplest case of the two-level density operators whose states are described in or on the Bloch sphere.

$$\hat{\rho} = \frac{1}{2} (\hat{1} + \vec{Q} \cdot \hat{\sigma}) \quad \text{where } \vec{Q} \text{ is Bloch vector}$$

$$0 \leq |\vec{Q}| \leq 1 \quad \begin{cases} \vec{Q} = 0 & \text{comp. mixed} \\ \vec{Q} = 1 & \text{pure} \end{cases}$$



$$\hat{\rho}_a = |\uparrow_a\rangle \langle \uparrow_a| \quad \hat{\rho}_b = |\uparrow_b\rangle \langle \uparrow_b|$$

$$\hat{\rho} = \frac{1}{2} |\uparrow_a\rangle \langle \uparrow_a| + \frac{1}{2} |\uparrow_b\rangle \langle \uparrow_b|$$

$$= \frac{3}{4} |\uparrow_z\rangle \langle \uparrow_z| + \frac{1}{4} |\downarrow_z\rangle \langle \downarrow_z|$$

Generally, need not have the same # of members in the ensemble

$$\vec{p} = \sum_i \lambda_i \vec{e}_i \quad \alpha \lambda_i < 1 \quad \sum_i \lambda_i = 1$$

$$\Rightarrow \hat{\rho} = \sum_i \lambda_i \underbrace{|\uparrow_i\rangle\langle\uparrow_i|}_{\frac{1}{2}(1 + \vec{e}_i \cdot \hat{\sigma})} \quad \text{"convex combination"}$$

Given this ambiguity, which different ensemble decompositions are equivalent for a general system (not just qubits).

Answer:

Suppose Two ensembles yield the same density operator

$$\left\{ \sum_{i=1}^N p_i |\psi_i\rangle \right\} \text{ and } \left\{ \sum_{j=1}^M q_j |\phi_j\rangle \right\}$$

iff:  $|\tilde{\psi}_i\rangle = \sum_{j=1}^M |\tilde{\phi}_j\rangle U_{ji}$

$$\text{where } |\tilde{\psi}_i\rangle = \sqrt{p_i} |\psi_i\rangle \quad \leftarrow \text{(not a normalized vector)}$$

$$|\tilde{\phi}_j\rangle = \sqrt{q_j} |\phi_j\rangle$$

And  $U_{ji}$  is a unitary matrix "padded" with extra zeros when  $N \neq M$

Here we show the proof in one direction.

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$$\text{Suppose } |\psi_i\rangle = \sum_{j=1}^M |\phi_j\rangle U_{ji}$$

$$\hat{\rho} = \sum_{i=1}^N p_i |\psi_i\rangle \langle \psi_i| = \sum_{i=1}^N |\psi_i\rangle \langle \psi_i|$$

$$= \sum_{j=1}^M \sum_{l=1}^M |\phi_j\rangle \underbrace{\sum_i U_{ji} U_{li}^*}_{\sum_i U_{ji} U_{il}^{-1} = \delta_{jl}} \langle \phi_l| = \sum_{j=1}^M |\phi_j\rangle \langle \phi_j|$$

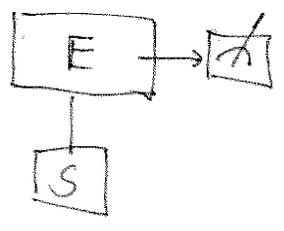
$$= \sum_{j=1}^M p_j |\phi_j\rangle \langle \phi_j| \quad \checkmark$$

For full proof ~~see~~ see text, "Quantum Computation and Quantum Information" by Nielsen & Chuang.

The non-uniqueness of the ensemble decomposition is a quantum property. "Classical probability" and "Quantum amplitudes" get combined.

This fact indicates why an unraveling of the evolving master equation into a set of pure quantum states is not unique.

The reason is that most measurements are done directly on the system being measured. Instead, the system is coupled to probe, and then the probe is measured.



For example, in the photodetector, the photons are coupled to electrons and then the photocurrent is measured.

This kind of indirect measurement is ubiquitous and calls for a more general theory of measurement.

This had been done long ago by the mathematical physicists, but is becoming very important today.

POVMs:

A set of operators  $\{ \hat{E}_\alpha \}$  defined a POVM if

- Eigenvalues of  $\hat{E}_\alpha$  are  $> 0$
- $\sum_\alpha \hat{E}_\alpha = \hat{I}$  resolution of identity

Measurement outcome  $P_\alpha = \text{Tr}(\hat{\rho} \hat{E}_\alpha)$

Post measurement state?

Indirect measurement  $\Rightarrow$  POVM's

In the standard picture of measurement, one considers an observable  $\hat{A}$  with eigenspectrum  $\{a, \hat{P}_a\}$

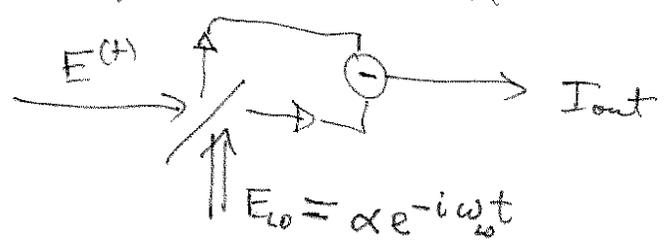
The probability of finding a

$$P_a = \text{Tr}(\rho^{(in)} \hat{P}_a)$$

Projector onto eigenspace space  
 $\sum_a \hat{P}_a = \hat{1}$

The post-measurement state  $\rho^{out} = \frac{\hat{P}_a \rho^{(in)} \hat{P}_a}{\text{Tr}(\rho^{in} \hat{P}_a)}$

Most measurements are not of this form. For example, for a photo-detector which counts photons the state of the field after detection of  $n$  photons is the vacuum, not the Fock state  $|n\rangle$ . Another example is heterodyne measurement



The output current  $I_{out} \sim \text{Tr}(\rho |\alpha\rangle\langle\alpha|)$

$$\int d^2\alpha \frac{|\alpha\rangle\langle\alpha|}{\pi} = \hat{1}$$

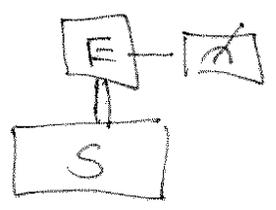
But  $\hat{E}_\alpha = \frac{|\alpha\rangle\langle\alpha|}{\pi}$  is not

a projector onto orthogonal subspace

$\{\hat{E}_\alpha\}$  is known as a "positive operator valued measure"  
 P O V M

# Indirect measurement as POVM

Consider now the bipartite system + "environment".  
 We want to find the measurement made on the system by coupling the two and then making a projective measurement on E.



We take the system ~~state~~ and environment to be initially uncoupled. At  $t=0$  the system state is arbitrary; the environment is taken in some fiducial state  $|0\rangle_E$

$$\Rightarrow \hat{\rho}_{SE}^{(in)} = \hat{\rho}_S^{(in)} \otimes |0\rangle_E \langle 0|$$

The two interact for some time, described by an entangling unitary  $\hat{U}_{SE}$

$$\Rightarrow \hat{\rho}_{SE}^{(out)} = \hat{U}_{SE} \hat{\rho}_{SE}^{(in)} \hat{U}_{SE}^\dagger$$

$$= \hat{U}_{SE} \left( \hat{\rho}_S^{(in)} \otimes |0\rangle_E \langle 0| \right) \hat{U}_{SE}^\dagger$$

Suppose we measure  $E$  according to some observable whose eigenstates (orthonormal) are  $\{| \mu_E \rangle\}$ .

The probability of find  $\mu$

$$\begin{aligned}
p_\mu &= \langle \mu_E | \hat{\rho}_E^{\text{out}} | \mu_E \rangle = \langle \mu_E | \text{Tr}_S (\hat{\rho}_{SE}^{\text{out}}) | \mu_E \rangle \\
&= \text{Tr}_S (\langle \mu_E | \hat{\rho}_{SE} | \mu_E \rangle) \\
&= \text{Tr}_S (\langle \mu_E | \hat{U}_{SE} | 0 \rangle_E \hat{\rho}_S^{\text{in}} \langle 0 | \hat{U}_{SE}^\dagger | \mu_E \rangle) \\
&= \text{Tr}_S (\hat{M}_\mu \hat{\rho}_S^{\text{in}} \hat{M}_\mu^\dagger) \\
&= \text{Tr}_S (\hat{E}_\mu \hat{\rho}_S^{\text{in}})
\end{aligned}$$

where  $\hat{E}_\mu \equiv \hat{M}_\mu^\dagger \hat{M}_\mu$  and  $\hat{M}_\mu \equiv \langle \mu_E | \hat{U}_{SE} | 0 \rangle_E$

POVM  $\sum_\mu \hat{E}_\mu = \sum_\mu \langle 0 | \hat{U}_{SE}^\dagger | \mu_E \rangle \langle \mu_E | \hat{U}_{SE} | 0 \rangle_E$

$$\begin{aligned}
&= \langle 0 | \hat{U}_{SE}^\dagger \hat{U}_{SE} | 0 \rangle_E \\
&= \hat{\mathbb{1}}_S \quad \checkmark
\end{aligned}$$

Thus, we have performed a measurement ~~on~~ environment and indirectly learned something about the system.

The post-measurement state

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$$\hat{\rho}_S^{\text{out}} |_{\mu} = \frac{\langle \mu_E | \hat{\rho}_{SE}^{\text{in}} | \mu_E \rangle}{\text{Tr}(\langle \mu_E | \hat{\rho}_{SE}^{\text{in}} | \mu_E \rangle)} = \frac{\hat{M}_{\mu} \hat{\rho}_S^{\text{in}} \hat{M}_{\mu}^{\dagger}}{p_{\mu}}$$

Now suppose we don't know the measurement result. Then we must sum over all possible results, weighted by their probabilities.

$$\hat{\rho}_S^{\text{out}} = \sum_{\mu} p_{\mu} \hat{\rho}_S^{\text{out}} |_{\mu} = \sum_{\mu} \hat{M}_{\mu} \hat{\rho}_S^{\text{in}} \hat{M}_{\mu}^{\dagger}$$

The map  $\hat{\rho}_S^{\text{in}} \Rightarrow \hat{\rho}_S^{\text{out}}$  is known as a super-operator. The representation

$$\hat{\rho}_S^{\text{out}} = \mathcal{E}[\hat{\rho}_S^{\text{in}}] = \sum_{\mu} \hat{M}_{\mu} \hat{\rho}_S^{\text{in}} \hat{M}_{\mu}^{\dagger}$$

is known as the "operator sum representation" or "Krause representation".

The operators  $\hat{M}_{\mu}$  are known as the Krause operators.

$$\{\hat{M}_{\mu}\}$$

The operator sum decomposition is an example of an ensemble preparation of  $\rho_S^{out}$ . The environment prepares the system and does not tell us the result.

Suppose  $\rho_S^{in} = |\psi_S^{in}\rangle\langle\psi_S^{in}|$  (a pure state)

$$\Rightarrow \rho_S^{out} = \sum_{\mu} p_{\mu} |\psi_{\mu}^{out}\rangle\langle\psi_{\mu}^{out}|$$

$$\text{where } |\psi_{\mu}^{out}\rangle = \frac{\hat{M}_{\mu} |\psi_S^{in}\rangle}{\sqrt{p_{\mu}}}$$

But we know the ensemble decomposition ~~is not~~ of  $\rho_S^{out}$  is not unique  $\Rightarrow$  the set of Krause operators  $\{\hat{M}_{\mu}\}$  which lead to the same superoperator is not unique

$$S[\rho_S^{in}] = \sum_{\mu} \hat{M}_{\mu} \rho_S^{in} \hat{M}_{\mu}^{\dagger} = \sum_{\nu} \hat{N}_{\nu} \rho_S^{in} \hat{N}_{\nu}^{\dagger}$$

$\hat{N}_{\nu} = \sum_{\mu} \hat{M}_{\mu} U_{\nu\mu}$   $\leftarrow$  Unitary matrix representing different basis for measuring  $E$