Lecture 29: The Stochastic Schrödinger Equation: Quantum State Diffusion

Jump processes:

We saw last lecture how the Linblad equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{1}{2}\sum_{\mu=1}^{m}\left(\hat{L}_\mu^\dagger \hat{L}_\mu \hat{\rho} + \hat{\rho}\hat{L}_\mu^\dagger \hat{L}_\mu - 2\hat{L}_\mu \hat{\rho}\hat{L}_\mu^\dagger\right)$$

generates a completely positive map

$$\hat{\rho}(t+dt) = \$_{dt}[\hat{\rho}(t)] = \sum_{\mu=0}^{m} \hat{M}_\mu(dt)\,\hat{\rho}(t)\,\hat{M}_\mu^\dagger(dt)$$

where the Krause operators are

$$\hat{M}_0 = \hat{\mathbb{1}} - \frac{1}{2}\sum_{\mu=1}^{m}\hat{L}_\mu^\dagger \hat{L}_\mu\, dt$$

$$\hat{M}_\mu = \hat{L}_\mu \sqrt{dt} \qquad \mu = 1, \ldots, m$$

This gives us a formal basis for quantum trajectories.

The probability of finding outcome μ is determined by POVM

$$P_\mu = \mathrm{Tr}(\hat{\rho}\,\hat{E}_\mu) \qquad \hat{E}_\mu = \hat{M}_\mu^\dagger \hat{M}_\mu$$

with post-measurement state $\quad \hat{\rho}\big|_\mu = \dfrac{\hat{M}_\mu \hat{\rho}\,\hat{M}_\mu^\dagger}{P_\mu}$

or for $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$

$$|\Psi(t+dt)\rangle\Big|_\mu = \frac{\hat{M}_\mu|\Psi(t)\rangle}{\sqrt{P_\mu}}$$

$$= \frac{\hat{L}_\mu|\Psi(t)\rangle}{\|\hat{L}_\mu|\Psi(t)\rangle\|}$$

Formally, we can define quantum trajectories using the "stochastic calculus" (See Gardiner's "Handbook of Stochastic Methods").

Given a state $|\Psi(t)\rangle$, we define a random variable, stochastic interval $dN_\mu(t)$, Poisson distributed with ~~expects~~ values

$$dN_\mu(t) = \begin{cases} 1 & \text{with probability } p_\mu \\ 0 & \text{with probability } 1-p_\mu \end{cases}$$

From these it clearly follows

$$\sum_{\mu=0}^{m} dN_\mu = 1$$

$$dN_\mu(t)\, dN_\nu(t) = \delta_{\mu\nu}\, dN_\mu(t)$$

$$\text{Expectation}\left(dN_\mu(t)\right) = p_\mu = \langle\Psi(t)|\hat{L}_\mu^\dagger \hat{L}_\mu|\Psi(t)\rangle\, dt$$

Then in unnormalized form

unnormalized

$$|\tilde{\Psi}(t+dt)\rangle = dN_0(t)\, \hat{M}_0|\Psi(t)\rangle + \sum_\mu dN_\mu(t)\, \hat{L}_\mu|\Psi(t)\rangle$$

$$= \hat{M}_0|\Psi(t)\rangle + \sum_{\mu=1}^{m} dN_\mu\left(\hat{L}_\mu - \hat{M}_0\right)|\Psi(t)\rangle$$

$$= \left(\mathbb{1} - \frac{i}{\hbar}\hat{H}_{eff}\, dt\right)|\Psi\rangle + \sum_{\mu=1}^{m} dN_\mu\left(\hat{L}_\mu - \left(\mathbb{1} - \frac{i}{\hbar}\hat{H}_{eff}\right)dt\right)|\Psi(t)\rangle$$

Now, according to the rules of Stochastic calculus

$dN_\mu \, dt = 0$ since dN_μ is nonzero on a set of measure zero

$$\therefore \; d|\tilde{\psi}\rangle = |\tilde{\psi}(t+dt)\rangle - |\tilde{\psi}(t)\rangle$$

$$\Rightarrow \boxed{d|\tilde{\psi}\rangle = -\frac{i}{\hbar}\hat{H}_{eff}\, dt\, |\psi\rangle + \sum_{\mu=1}^{m} dN_\mu(t)\,(\hat{L}_\mu - 1)\,|\psi\rangle}$$

In is the Stochastic schrodinger equation for a "jump process", in unnormalized form.

Note: Though it appears linear, it is not since the probability $dN_\mu(t)$ takes on value 0 or 1 depends on $|\psi(t)\rangle$

In Normalized form:

$$\boxed{\begin{aligned} d|\psi\rangle &= \left(-\frac{i}{\hbar}\hat{H}_{eff} + \frac{1}{2}\sum_{\mu=1}^{m}\langle \hat{L}_\mu^{\dagger}\hat{L}_\mu\rangle\right) dt\,|\psi\rangle \\ &+ \sum_{\mu=1}^{m} dN_\mu(t)\left(\frac{\hat{L}_\mu}{\sqrt{\langle \hat{L}_\mu^{\dagger}\hat{L}_\mu\rangle}} - 1\right)|\psi\rangle \end{aligned}}$$

explicitly nonlinear

Let us show that formally the S.S.E., when averaged, yields the Master eqn. (29.4)

$$|\psi(t+dt)\rangle = |\psi(t)\rangle + d|\psi\rangle$$

$$\hat{\rho}(t+dt) = \overline{|\psi(t+dt)\rangle\langle\psi(t+dt)|} = \hat{\rho}(t) + \overline{|\psi\rangle\langle d\psi|} + \overline{|d\psi\rangle\langle\psi|} + \overline{|d\psi\rangle\langle d\psi|}$$

$$\Rightarrow d\hat{\rho} = -\frac{i}{\hbar}[\hat{H}_{eff}, \hat{\rho}]dt + \sum_{\mu=1}^{m} \langle \hat{L}_\mu^\dagger \hat{L}_\mu \rangle dt \, \hat{\rho} + \sum_{\mu=1}^{m} \overline{dN_\mu(t)} \left(\frac{\hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger}{\langle \hat{L}_\mu^\dagger \hat{L}_\mu \rangle} - \hat{\rho} \right)$$

Given $\overline{dN_\mu} = P_\mu = \langle \hat{L}_\mu^\dagger \hat{L}_\mu \rangle dt$

$$\Rightarrow \frac{d\rho}{dt} = \frac{-i}{\hbar}[\hat{H}_{eff}, \hat{\rho}]' + \sum_{\mu=1}^{m} \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger$$

The Linblad form of the Master eqn.

Continuous S.S.E.

We have seen that an equivalence class of "unravellings" of the Master eqn can be obtained for a unitary remixing of the Krause operators $\hat{M}_\mu = \hat{L}_\mu \sqrt{dt}$ $\mu = 1, \ldots, m$.

This is a limited case. A more general equivalence class is found, including $\hat{M}_0 = \hat{\mathbb{1}} - \frac{i}{\hbar} \hat{H}_{eff}\, dt$.

Consider a new set of Linblad operators

$$\hat{J}_{\mu,\pm} = \frac{A_\mu \hat{\mathbb{1}} \pm \hat{L}_\mu}{\sqrt{2}}$$ where A_μ is a complex constant.

Straight forward algebra shows that these generate the same master eqn., i.e.,

$$\mathcal{L}_{relax}[\hat{\rho}] = -\frac{1}{2}\sum_\mu \left(\hat{L}_\mu^\dagger \hat{L}_\mu \hat{\rho} + \hat{\rho} \hat{L}_\mu^\dagger \hat{L}_\mu - 2 \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger \right)$$

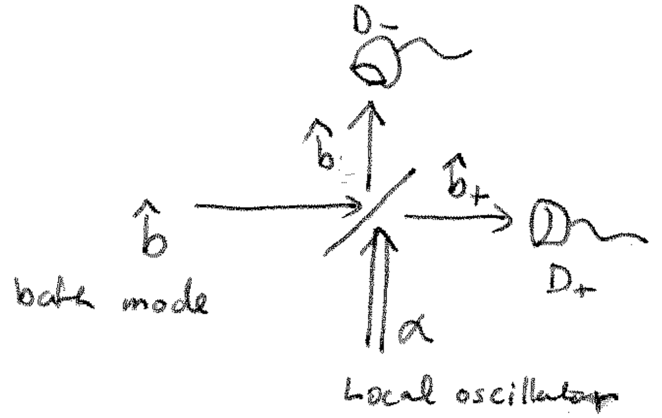
$$= -\frac{1}{2}\sum_{\mu, \epsilon = \pm} \left(\hat{J}_{\mu,\epsilon}^\dagger \hat{J}_{\mu,\epsilon} \hat{\rho} + \hat{\rho} \hat{J}_{\mu,\epsilon}^\dagger \hat{J}_{\mu,\epsilon} - 2 \hat{J}_{\mu,\epsilon} \hat{\rho} \hat{J}_{\mu,\epsilon}^\dagger \right)$$

These can be seen as the unitary remixing of Krause ops:

$$\begin{bmatrix} \hat{K}_0 \\ \hat{K}_{\mu,+} \\ \hat{K}_{\mu,-} \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}|A_\mu|^2 dt & 0 & A_\mu \sqrt{\frac{dt}{2}} \\ A_\mu \sqrt{\frac{dt}{2}} & \frac{1}{\sqrt{2}} & \frac{1 - \frac{1}{2}|A_\mu|^2 dt}{\sqrt{2}} \\ A_\mu \sqrt{\frac{dt}{2}} & -\frac{1}{\sqrt{2}} & \frac{1 - \frac{1}{2}|A_\mu|^2 dt}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \hat{M}_0 \\ \hat{M}_\mu \\ 0 \end{bmatrix}$$

We recognize the new jump operators physically.

Recall homodyne detection:



$$\hat b = \frac{\hat{\mathbb{1}}\alpha + \hat b}{\sqrt{2}}$$

$$\hat b = \frac{\hat{\mathbb{1}}\alpha - \hat b}{\sqrt{2}}$$

(for some choice of beam splitter phase)

The new jump operator $\hat J_{\mu,\pm}$ represent the effect on the system when the bath modes are detected as $D_\pm$.

The unnormalized stochastic Schrödinger eqn. now takes the form

$$d|\tilde\psi\rangle = \left(-\frac{i}{\hbar}\hat H_{eff}\,dt + \sum_{\mu,\epsilon=\pm} dN_{\mu,\epsilon}\left(\hat J_{\mu\epsilon} - 1\right)\right)|\psi\rangle$$

with
$$\hat H_{eff} = \hat H - \frac{1}{2}\sum_{\mu,\epsilon=\pm} \hat J^\dagger_{\mu,\epsilon}\hat J_{\mu\epsilon}$$

$$= \hat H - \frac{1}{2}\Big(\underbrace{|\alpha|^2}_{\text{irrelevant for unnormalized state}} + \sum_\mu \hat L^\dagger_\mu \hat L_\mu\Big)$$

Now the probability of seeing a jump

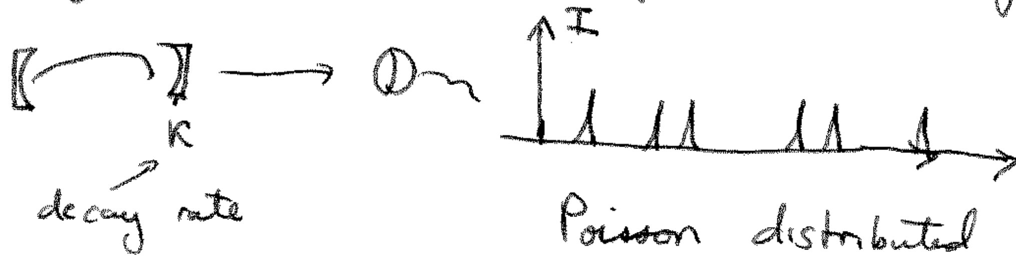
$$P_{\mu,\pm} = dt \langle \hat{J}^{\dagger}_{\mu,\pm} \hat{J}_{\mu,\pm} \rangle = \frac{1}{2} |A_\mu|^2 dt \pm \frac{1}{2} |A_\mu| \langle \hat{X}_\mu(\phi) \rangle dt + \frac{1}{2} \langle \hat{L}^{\dagger}_\mu \hat{L}_\mu \rangle dt$$

where $A_\mu = |A_\mu| e^{i\phi}$, $\hat{X}_\mu(\phi) = \hat{L}_\mu e^{-i\phi} + \hat{L}^{\dagger}_\mu e^{i\phi}$.
(phase quadrature)

When the local oscillator is "macroscopic", the rate of photodetections is <u>huge</u>. Practically speaking, a numerical simulation would require $|A|^2 dt \ll 1$ so dt would need to be incredibly small. On the other hand, most of these detections are arising from the local oscillator, not the spontaneous emission from the system, so their effect is small. ~~thus~~ Moreover, the detection has more of the character of a <u>continuous photocurrent</u> rather than <u>discrete photocounts</u>.

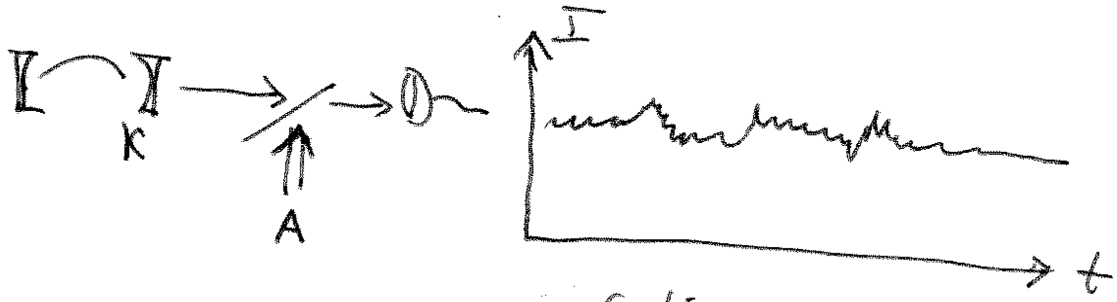
Schematically, consider the decay of a cavity

Direct



Poisson distributed counts

Homodyn



Continuous signal with noise

Local oscillator flux $|A|^2$. Macroscopic $\Rightarrow$ # of photons in time $\frac{1}{\kappa} \gg 1$. Define $A = \sqrt{\kappa}\,\alpha \Rightarrow |\alpha|^2 \gg 1$

In order to make the transaction from the discrete Poisson process to the continuous noise we "coarse grain" the quantum trajectory. When consider a chunk of time δt such that

$$\frac{1}{|A|^2} = \frac{1}{|\alpha|^2 \kappa} \ll \delta t \ll \frac{1}{\kappa}$$

In that time many individual photodetections have occurs, but the state has changed very little and can thus be taken as a different w.r.t. the system.

Taking $\frac{1}{|\alpha|}$ as the small parameter, over a time δt many jumps have occured in detectors $D_\pm$ and the unnormalized state evolves as:

$$|\tilde{\Psi}(t+\delta t)\rangle = e^{-iH_{eff}\,\delta t/\hbar} \left(\hat{J}_{\mu,+}\right)^{N_{\mu,+}} \left(\hat{J}_{\mu,-}\right)^{N_{\mu,-}} |\Psi(t)\rangle$$

where $N_{\mu,\pm}$ are the number of counts in $D_\pm$

Note: The jump and non-jump ops commute to $\mathcal{O}(\epsilon)$

Aside $\left(\hat{J}_{\mu,+}\right)^{N_{\mu,+}} \left(\hat{J}_{\mu,-}\right)^{N_{\mu,-}} = \underbrace{\left(\frac{A_\mu}{\sqrt{2}}\right)^N}_{\text{irrelevant for unnormalized state}} \left(\mathbb{1} + \frac{1}{A_\mu} \sum_\mu (N_{\mu,+} - N_{\mu,-})\,\hat{L}_\mu\right)$

~~to order~~ to lowest order

where $N = \sum_{\mu,\sigma} N_{\mu,\sigma}$

$$\Rightarrow |\tilde{\Psi}(t+\delta t)\rangle = \left[\left(\mathbb{1} - \frac{i}{\hbar}\hat{H}_{eff}\right)\delta t + \sum_\mu (N_{\mu,+} - N_{\mu,-})\,\frac{\hat{L}_\mu}{A_\mu}\right] |\tilde{\Psi}(t)\rangle$$

This equation is still of the familiar form.

Now $N_{\mu,\pm}$ are random variables, poisson distributed. In the limit of large mean, the Poisson distribution is approximated by a continuous Gaussian (by the central limit theorem). This Gaussian describes a <u>Weiner process</u> (Brownian motion / diffusion)

$$N_{\mu,\epsilon}(t) = \overline{N_{\mu,\epsilon}(t)} + (\Delta N_{\mu,\epsilon})\, \delta W_{\mu,\epsilon}(t)$$

mean — Variance — Weiner stochastic variable (Gaussian white noise)

$$\overline{\delta W_{\mu\epsilon}^{\,2}} = \delta t$$

Here

$$\overline{N_{\mu,\epsilon}(t)} = \frac{|A|^2 \delta t}{2}\left(1 + \frac{\epsilon}{|A|}\langle \hat{X}_{\mu}(\phi)\rangle\right)$$

$$\Delta N = \sqrt{\overline{N}} \quad \text{(Poisson fluctuation)}$$

In our case only the difference current matters (balance homodyne)

$$\Rightarrow N_{\mu,+} - N_{\mu,-} = |A|\delta t \langle \hat{X}_{\mu}(\phi)\rangle + |A|\,\delta W_{\mu}$$

$$= \text{Balance homodyne current} = I_{\mu}(\phi)\, \delta t$$

$$I_{\mu}(\phi) = |A|\left(\langle \hat{X}_{\mu}(\phi)\rangle + \xi_{\mu}\right)$$ Gaussian white noise

The unnormalize S.S.E. then takes the form of a "Langevin equation"

$$\boxed{\begin{aligned} d|\tilde{\psi}\rangle &= \left[-\frac{i}{\hbar} H_{eff}\, dt + \sum_\mu \left(\langle \hat{X}_\mu(\phi)\rangle dt + dW_\mu\right) \hat{L}_\mu\right] |\tilde{\psi}\rangle \\ \text{where} \quad H_{eff} &= H - \frac{1}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu \\ \hat{X}_\mu(\phi) &= \hat{L}_\mu e^{-i\phi} + \hat{L}_\mu^\dagger e^{i\phi} \\ dW_\mu &= \text{Weiner increment} \end{aligned}}$$

Sometimes this is written so as to explicitly show the conditional evolution due to the homodyne current:

$$\boxed{\begin{aligned} \frac{d}{dt}|\tilde{\psi}\rangle &= \left[\left(-\frac{i}{\hbar}\hat{H} - \frac{1}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu\right) + \sum_\mu \frac{I_\mu(\phi)}{|A_\mu|}\hat{L}_\mu\right]|\tilde{\psi}\rangle \\ I_\mu(\phi) &= |A_\mu|\left(\langle \hat{X}_\mu(\phi)\rangle + \xi_\mu\right) \end{aligned}}$$

29.12

When normalized the homodyne S.S.E. takes the form

$$\boxed{d|\psi\rangle = \Big\{\Big(-\frac{i}{\hbar}H_{eff} - \sum_\mu \frac{\langle \hat{X}_\mu\rangle}{2}\hat{L}_\mu + \frac{1}{8}\langle \hat{X}_\mu\rangle^2\Big)\Big)dt + \sum_\mu \Big(\hat{L}_\mu - \frac{1}{2}\langle \hat{X}_\mu\rangle\Big) dW_\mu \Big\}|\psi\rangle}$$

Let us confirm that this is unravelling of the master eqn.

$$\hat{\rho}(t+dt) = \overline{(|\psi(t)\rangle + |d\psi\rangle)(\langle\psi(t)| + \langle d\psi|)}$$

$$\Rightarrow d\hat{\rho} = \overline{|d\psi\rangle\langle\psi|} + \overline{|\psi\rangle\langle d\psi|} + \overline{|d\psi\rangle\langle d\psi|}$$

$$= -\frac{i}{\hbar}[H_{eff}, \hat{\rho}]' dt - \sum_\mu \Big[\frac{1}{4}\langle \hat{X}_\mu\rangle^2\hat{\rho} + \frac{\langle \hat{X}_\mu\rangle}{2}(\hat{L}_\mu\hat{\rho} + \hat{\rho}\hat{L}_\mu^\dagger)\Big]dt$$

$$+ \sum_\mu \Big[\hat{L}_\mu\hat{\rho}\hat{L}_\mu^\dagger + \frac{1}{4}\langle \hat{X}_\mu\rangle^2\hat{\rho} - \frac{1}{2}\langle \hat{X}_\mu\rangle(\hat{L}_\mu\hat{\rho} + \hat{\rho}\hat{L}_\mu^\dagger)\Big]\overline{dW_\mu^2}$$

$$\overline{dW_\mu^2} \Downarrow dt$$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[H_{eff}, \hat{\rho}]' + \sum_\mu \hat{L}_\mu\hat{\rho}\hat{L}_\mu^\dagger \quad ✓$$

An alternative derivation of the continuous S.S.E can be seen as follows. Consider the S.S.E in normalized form

$$\frac{d|\psi\rangle}{dt} = \left[-\frac{i}{\hbar}\hat{H} + \sum_{\mu} \left\{ \frac{-\hat{J}^{\dagger}_{\mu\epsilon}\hat{J}_{\mu\epsilon} + \langle \hat{J}^{\dagger}_{\mu,\epsilon}\hat{J}_{\mu,\epsilon}\rangle}{2} + \left(\frac{\hat{J}_{\mu,\epsilon}}{\sqrt{\langle \hat{J}^{\dagger}_{\mu,\epsilon}\hat{J}_{\mu\epsilon}\rangle}} - 1 \right) \frac{dN_{\mu,\epsilon}}{dt} \right\} \right] |\psi\rangle$$

The quantities $I_{\mu,\epsilon} \equiv \frac{dN_{\mu,\epsilon}}{dt}$ are the stochastic currents corresponding to the various "detectors". For jump process $\frac{dN_{\mu,\epsilon}}{dt}$ represents Poisson distributed discrete events. For Weiner process $\frac{dN_{\mu,\epsilon}}{dt}$ represents Gaussian distributed noise about some mean value:

$$\frac{dN_{\mu,\epsilon}}{dt} = \overline{I}_{\mu\epsilon}(t) + \sqrt{\overline{I}_{\mu\epsilon}}\,\xi_{\mu,\epsilon}(t)$$

Gaussian White noise

Here $\hat{J}_{\mu,\epsilon} = \frac{A_\mu + \epsilon \hat{L}_\mu}{\sqrt{2}}$ $\qquad \hat{J}^{\dagger}_{\mu\epsilon}\hat{J}_{\mu\epsilon} = \frac{1}{2}\left(|A_\mu|^2 + \epsilon |A_\mu| \hat{X}_\mu(\phi) + \hat{L}^{\dagger}_\mu \hat{L}_\mu\right)$

and $\overline{I}_{\mu\epsilon} = \langle \hat{J}^{\dagger}_{\mu\epsilon}\hat{J}_{\mu\epsilon}\rangle \cong \frac{1}{2}\left(|A_\mu|^2 + \epsilon |A_\mu| \langle \hat{X}_\mu(\phi)\rangle\right)$ for large $|A_\mu|$

$$\frac{dN_{\mu\epsilon}}{dt} \approx \frac{1}{2}\left(|A_\mu|^2 + \epsilon |A_\mu| \langle \hat{X}_\mu(\phi)\rangle\right) + \frac{1}{\sqrt{2}}|A_\mu|\,\xi_{\mu\epsilon}(t)$$

Plugging these definitions in and keep terms up to $\mathcal{O}(1)$ in $\frac{1}{|A_\mu|}$

$$\Rightarrow \boxed{\begin{aligned} d|\psi\rangle = \Big\{ &\Big(-\frac{i}{\hbar}\hat{H} + \sum_\mu \Big(-\frac{1}{2}\hat{L}_\mu^\dagger \hat{L}_\mu + \langle \hat{X}_\mu(\phi)\rangle \hat{L}_\mu - \frac{\langle \hat{X}_\mu\rangle^2}{4}\Big)\Big)dt \\ &+ \Big(\hat{L}_\mu - \frac{1}{2}\langle \hat{X}_\mu(\phi)\rangle\Big) dW_\mu \Big\} |\psi\rangle \\ \text{where } \; dW_\mu = \xi_\mu\, dt \qquad & \xi_\mu = \frac{1}{\sqrt{2}}(\xi_{\mu+} + \xi_{\mu-}) \end{aligned}}$$

As before:

<u>Application:</u> Driven two-level atom seen as diffusion on the Bloch sphere

Consider our favorite problem: A 2-lev atom driven by a classical laser field and coupled to the vacuum.

then $\hat{H} = -\frac{\hbar\Delta}{2}\hat{\sigma}_z - \frac{\hbar\Omega}{2}\hat{\sigma}_x$ in R.W.A.

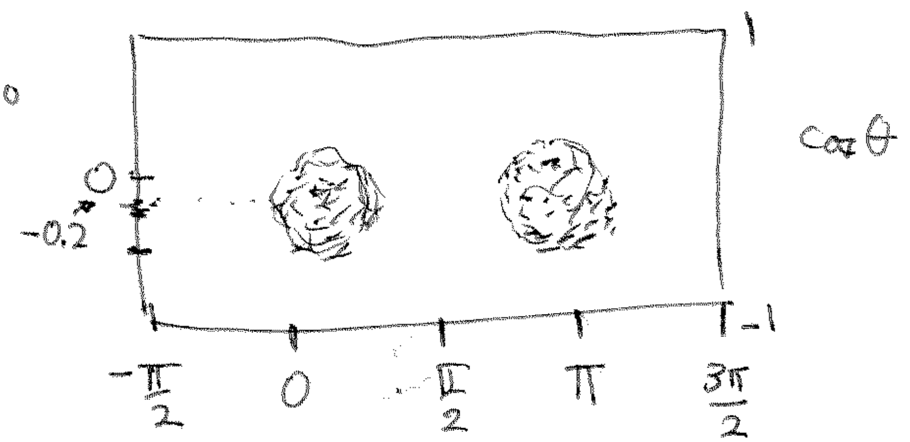
$\hat{L} = \sqrt{\Gamma}\,\hat{\sigma}_-$ jump operator

A simulation using the "homodyne" S.S.E was carried by Wiseman and Milburn (PRA 47 1652 1993).

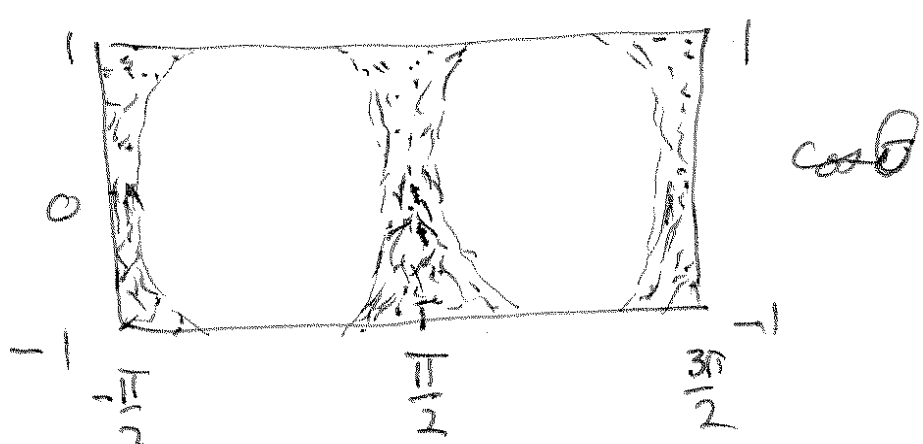
Choosing a laser detuning $\Delta = 0$ (resonance) and $\Omega^2 = 14\Gamma^2$ (highly saturated), the steady state solution has a Bloch vector $|\vec{Q}| \approx 0.1$ with $\cos\theta \approx -0.2$, $\phi = \frac{\pi}{2}$

The steady state distribution of Bloch vectors for many simulations is sketched below for two choices of the homodyne phase $\phi_H = 0$, $\phi_H = \frac{\pi}{2}$

$\phi_H = 0$



$\phi_H = \frac{\pi}{2}$



The distributions yield the same mean values, as they must, but are very different in nature. This, of course reflects the very different nature of the information obtained by the homodyne measurements.

For $\phi_H = 0$, one is measuring the radiated field in-phase with the laser. This is the $u = \langle \hat{\sigma}_x \rangle$ component of the Bloch vector. Such measurements tend to localize the state ~~near~~ near the eigenstates $|\pm_x\rangle$, which are the caps near $\theta = 0$, ~~π~~ $\phi = 0, \pi$. Since these are also eigenstates of the atom-laser interaction they remain localized there. The detailed balance between the laser + spontaneous emission pushes the steady state below the equator on the Bloch sphere and toward $\phi = \pi/2$.

In contrast for $\phi_H = \pi/2$, one ~~is~~ measuring the in-quadrature component of the dipole oscillation $v = \langle \hat{\sigma}_y \rangle$. $|\pm_y\rangle$ states are not eigenstates of the laser and rapidly oscillate along great circles at constant ϕ.

Heterodyne: Suppose one does not fix the phase of the local oscillator, but instead allows it to rotate rapidly in the phase plane. This is known as heterodyne detection, and is accomplished by choosing the local oscillator at a different frequency than the signal. The phase then oscillates at the beat note $\Omega = \omega_{LO} - \omega_{signal}$ (see Walls and Milburn).

Whereas homodyne measures a hermitian operator, the quadrature $\hat{X}_\phi$, Heterodyne measures the non-Hermitian operator $\hat{b}_\mu^\dagger$, i.e. a P.O.V.M for path mode μ.

From the unnormalized form of the S.S.E on page 29.11 we obtain

$$d|\tilde{\psi}\rangle = \left[-\frac{i}{\hbar} H_{eff}\, dt + \sum_\mu \left(\langle \hat{L}_\mu^\dagger \rangle + dW_\mu \right) \hat{L}_\mu \right] |\tilde{\psi}\rangle$$

or in normalized form,

$$d|\psi\rangle = \left[-\frac{i}{\hbar} H\, dt + \sum_\mu \left(\langle \hat{L}_\mu^\dagger \rangle \hat{L}_\mu - \frac{1}{2} \hat{L}_\mu^\dagger \hat{L}_\mu - \frac{1}{2} |\langle \hat{L}_\mu \rangle|^2 \right) dt + \sum_\mu \left(\hat{L}_\mu - \langle \hat{L}_\mu \rangle \right) \right] |\psi\rangle$$

This form of the S.S.E. was introduced by Gisin and Percival (PRL 52 1657 (1984); Helvetica Phys. Act 62, 363 (1989); Phys. Lett A 143, 1 (1990); J. Phys. A. 25, ~~58~~77 (1992); Phys. Let A 175 144 (1993))

Termed "Quantum State Diffussion" it was seen as a way of modeling quantum measurement as a <u>dynamical process</u> rather than an ad hoc collapse. Today we understand it as one possible unravelling of the master equation conditioned on heterodyne measurement.

When $\hat{L}_\mu$ is Hermitian (~~set~~ one jump operator)

$$d|\psi\rangle = \left\{\left(-\frac{i}{\hbar}H - \frac{1}{2}\Delta\hat{L}^2\right)dt + \Delta\hat{L}\,dW\right\}|\psi\rangle$$

$$\text{where } \Delta\hat{L} = \hat{L} - \langle\hat{L}\rangle$$

This equation clearly involves a "drift term", pushing $\langle\hat{L}\rangle$ to it's expection value, plus diffussion.