

Dynamics of Two Qubits: Purity Swapping and an Entanglement Optimization Protocol

Cesar A. Rodriguez,* Anil Shaji, and E. C. G. Sudarshan

The University of Texas at Austin, Center for Statistical Mechanics, 1 University Station C1602, Austin TX 78712

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The evolution of two qubits coupled by a general nonlocal interaction is studied. The dynamical map for the reduced evolution of one qubit is computed. We look at two distinct regimes which depend on the choice of parameters for the interaction and the initial state. In the first, the purity of the individual qubits is interchanged with the entanglement shared by the two. In the second regime, the interaction entangles two initially pure qubits. We outline a protocol for optimizing the entanglement generation.

Entanglement is considered a *resource* in quantum information theory [1, 2]. Naturally, there has been a lot of interest¹ in understanding, quantifying and controlling it. This is clear from the numerous efforts towards defining a measure of entanglement [3, 4, 5, 6, 7], as well as the broad range of experiments [8, 9, 10] exploring ways to tame it. Nevertheless, the smallest Hilbert Space in which there can be entanglement is not fully understood analytically. Some numerical efforts have started to scratch the surface [11]. Two-qubit entanglement allows operations such as entanglement swapping [12], purification [13] and teleportation [2]. Processes that create entanglement have the practical limitation that they have to compete against decoherence phenomena. Therefore, optimizing such processes is of importance, and in reference [14] a method has been proposed where, given some entangling Hamiltonian, local operations on each qubit can be done to maximize the entanglement rate. This two-qubit space is of significant interest in the context of building a quantum computer. Universal quantum gates, that can serve as the building blocks of such a computer, are two-qubit operations [15, 16].

In this Letter we study the most general interaction Hamiltonian that couples two qubits. Our main result consists of an explicit expression for the dynamics of this system that lets us study various features of the two-qubit space. We then focus on how the dynamics affect the purity of the individual qubits through the entanglement between them. An example is purity swapping, where one qubit is dynamically purified at the expense of another. This may be contrasted with entanglement swapping [17]. We also study the generation of entanglement using Hamiltonian dynamics, leading to a protocol that optimizes any given entangling process subject to reasonable constraints.

The most general dynamics of the quantum state represented by a density matrix ρ can be described in terms of a dynamical map [18, 19]:

$$\rho \rightarrow \mathfrak{B}(\rho). \quad (1)$$

This formalism has proven to be very useful in understanding the dynamics of open quantum systems. The dynamical map represents the effect of the coupled unitary evolution of the system and its environment [20]. In other words:

$$\rho^A \rightarrow \rho^A(t) = Tr_B [U \rho^A \otimes \rho^B U^\dagger] = \mathfrak{B} \rho^A, \quad (2)$$

where ρ^A is the state of the system and ρ^B is that of the environment. Here we have assumed that the system and the environment start in a separable state, $\rho^A \otimes \rho^B$.

We concentrate on the simplest example, where both ρ^A and ρ^B are single qubits:

$$\rho^A = \frac{\mathbb{1}^A + \sum_i a_i \sigma_i^A}{2}, \quad \rho^B = \frac{\mathbb{1}^B + \sum_i b_i \sigma_i^B}{2}, \quad (3)$$

with σ_i^A and σ_i^B being the Pauli spin matrices for each of them. The Bloch vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ provide a convenient way of parameterizing single qubit states that we will use in this Letter. Together, ρ_A and ρ_B form the initially separable 4×4 state,

$$E^{AB}(0) = \rho^A \otimes \rho^B = \frac{1}{4} [\mathbb{1}^A \otimes \mathbb{1}^B + \sum_i (a_i \sigma_i^A \otimes \mathbb{1}^B + b_i \mathbb{1}^A \otimes \sigma_i^B) + \sum_{i,j} a_i b_j \sigma_i^A \otimes \sigma_j^B], \quad (4)$$

where subscripts take values from $\{1, 2, 3\}$. The most general Hamiltonian for two qubits is:

$$H = \sum_i \alpha_i \sigma_i^A \otimes \mathbb{1}^B + \sum_i \beta_i \mathbb{1}^A \otimes \sigma_i^B + \sum_{i,j} \Gamma_{ij} \sigma_i^A \otimes \sigma_j^B. \quad (5)$$

In the interaction picture, it becomes,

$$H \rightarrow H(t) = \sum_{i,j} \tilde{\Gamma}_{ij}(t) \sigma_i^A \otimes \sigma_j^B, \quad (6)$$

which has nine parameters. With suitable local unitary transformations with three parameters each, that act on

¹ “As the strong man exults in his physical ability, delighting in such exercises as call his muscles into action, so glories the analyst in that moral activity which *disentangles*.” [sic] Poe, E.A. “The Murders in the Rue Morgue” *Complete Stories and Poems of Edgar Allan Poe*. Ed. Doubleday & Company, Inc. Garden City, NY (1966)

each one of the qubits, the number can be brought down to three [20, 21, 22]:

$$H(t) \rightarrow H(t) = \sum_i \gamma_i(t) \sigma_i^A \otimes \sigma_i^B. \quad (7)$$

The time evolution of the overall state E^{AB} is given by,

$$E^{AB}(t) = U E^{AB}(0) U^\dagger, \quad (8)$$

where $U = e^{-i \int H(t) dt}$. For simplicity, we will assume that there is no free evolution for individual qubits, making $\gamma_i(t) \rightarrow \gamma_i$, and,

$$U \rightarrow U = \exp(-iHt) = \prod_{j=1}^3 [\cos(2\gamma_j t) \mathbb{1}^A \otimes \mathbb{1}^B - i \sin(2\gamma_j t) \sigma_j^A \otimes \sigma_j^B]. \quad (9)$$

To calculate $E^{AB}(t)$, use the property that $\sigma_1^A \otimes \sigma_1^A, \sigma_2^B \otimes \sigma_2^B, \sigma_3^A \otimes \sigma_3^B$ all commute with each other. For each of the terms of $E^{AB}(0)$ we obtain:

$$\begin{aligned} U \mathbb{1} \otimes \mathbb{1} U^\dagger &= \mathbb{1} \otimes \mathbb{1}, \\ U \sigma_i \otimes \sigma_i U^\dagger &= \sigma_i \otimes \sigma_i, \\ U \mathbb{1} \otimes \sigma_i U^\dagger &= \mathbb{1} \otimes \sigma_i e^{2i(\gamma_j \sigma_j \otimes \sigma_j + \gamma_k \sigma_k \otimes \sigma_k)}, \\ U \sigma_i \otimes \sigma_j U^\dagger &= \sigma_i \otimes \sigma_j e^{2i(\gamma_i \sigma_i \otimes \sigma_i + \gamma_j \sigma_j \otimes \sigma_j)}, \end{aligned} \quad (10)$$

Using Eq. (10), the evolution generated by Eq. (9) of the two qubit density matrix is:

$$\begin{aligned} E^{AB}(t) &= \frac{1}{4} \sum_{i=1}^3 [\mathbb{1}^A \otimes \mathbb{1}^B + a_i (C_j C_k \sigma_i^A \otimes \mathbb{1}^B + S_j S_k \mathbb{1}^A \otimes \sigma_i^B + C_k S_j \sigma_k^A \otimes \sigma_j^B - C_j S_k \sigma_j^A \otimes \sigma_k^B) \\ &\quad + b_i (C_j C_k \mathbb{1}^A \otimes \sigma_i^B + S_j S_k \sigma_i^A \otimes \mathbb{1}^B + C_k S_j \sigma_j^A \otimes \sigma_k^B - C_j S_k \sigma_k^A \otimes \sigma_j^B) + a_i b_i \sigma_i^A \otimes \sigma_i^B \\ &\quad + a_i b_j (C_i C_j \sigma_i^A \otimes \sigma_j^B + S_i S_j \sigma_j^A \otimes \sigma_i^B + C_i S_j \sigma_k^A \otimes \mathbb{1}^B - C_j S_i \mathbb{1}^A \otimes \sigma_k^B) \\ &\quad + a_j b_i (C_i C_j \sigma_j^A \otimes \sigma_i^B + S_i S_j \sigma_i^A \otimes \sigma_j^B + C_i S_j \mathbb{1}^A \otimes \sigma_k^B - C_j S_i \sigma_k^A \otimes \mathbb{1}^B)], \end{aligned} \quad (11)$$

where $C_i \equiv \cos(2t\gamma_i)$, $S_i \equiv \sin(2t\gamma_i)$ and i, j, k are cyclic. In other words, the coefficients of the Pauli matrices in Eq. (4) transform as follows:

$$\begin{aligned} \begin{Bmatrix} a_k \\ b_k \end{Bmatrix} &\rightarrow \begin{Bmatrix} a_k \\ b_k \end{Bmatrix} C_i C_j + \begin{Bmatrix} b_k \\ a_k \end{Bmatrix} S_i S_j \\ &+ \begin{Bmatrix} a_i b_j \\ a_j b_i \end{Bmatrix} C_i S_j - \begin{Bmatrix} a_j b_i \\ a_i b_j \end{Bmatrix} C_j S_i, \end{aligned} \quad (12)$$

where $\{i, j, k\}$ are cyclic, and also as:

$$\begin{aligned} a_i b_j &\rightarrow a_i b_j C_i C_j + a_j b_i S_i S_j \\ &+ \epsilon_{ijk} b_k C_j S_i - \epsilon_{ijk} a_k C_i S_j, \end{aligned} \quad (13)$$

where it is not required for $\{i, j, k\}$ to be cyclic or distinct. Eqs. (12,13) may be extended in a straightforward fashion to the case where the initial state, \tilde{E} , is not simply separable, i.e.:

$$\tilde{E} = \frac{1}{4} \sum_{i,j} [\mathbb{1} \otimes \mathbb{1} + e_{i0} \sigma_i \otimes \mathbb{1} + \quad (14)$$

$$e_{0i} \mathbb{1} \otimes \sigma_i + e_{ij} \sigma_i \otimes \sigma_j], \quad (15)$$

where $e_{ij} \neq e_{i0} \times e_{0j}$.

To find the reduced dynamics of the system, ρ^A , we just need to carry out the partial trace from Eq. (2).

Using Eq. (11), Eqs. (12,13) and the fact that σ_i^B are traceless, this is quite straightforward:

$$\rho^A(t) = \frac{1}{2} \begin{pmatrix} 1 + a_3(t) & a_1(t) - ia_2(t) \\ a_1(t) + ia_2(t) & 1 - a_3(t) \end{pmatrix}, \quad (16)$$

where:

$$\begin{aligned} a_i(t) &= a_i C_j C_k + b_i S_j S_k \\ &+ a_j b_k C_j S_k - a_k b_j C_k S_j. \end{aligned} \quad (17)$$

The purity of the reduced density matrix changes with time:

$$\begin{aligned} P^A(t) &= Tr [\rho^A(t)^2] \\ &= \frac{1}{2} [1 + a_1(t)^2 + a_2(t)^2 + a_3(t)^2]. \end{aligned} \quad (18)$$

Following a similar procedure we can find the evolution of $\rho^B(t)$ and its purity $P^B(t)$.

The time evolution of ρ^A can be described using the dynamical map from Eq. (1). The mapping matrix can be explicitly calculated to be [18]:

$$\mathfrak{B}^{(0,t)} = \frac{1}{2} \begin{pmatrix} 1 + b_3 S_1 S_2 + C_1 C_2 & b_2 C_1 S_2 - i b_1 C_2 S_1 & (b_1 S_3 - b_2 C_3) S_2 & (C_1 + C_2) \\ b_2 C_1 S_3 + i b_1 C_2 S_1 & 1 + b_3 S_1 S_2 - C_1 C_2 & -i (b_2 S_3 + b_1 C_3) S_1 & \times (C_3 + i b_3 S_3) \\ (b_1 S_3 - b_2 C_3) S_2 & (C_2 - C_1) & \times (C_3 - i b_3 S_3) & (b_1 S_3 + b_2 C_3) S_2 \\ +i (b_2 S_3 + b_1 C_3) S_1 & \times (C_3 + i b_3 S_3) & 1 - b_3 S_1 S_2 - C_1 C_2 & -i (b_2 S_3 - b_1 C_3) S_1 \\ (C_1 + C_2) & (b_1 S_3 + b_2 C_3) S_2 & -b_2 C_1 S_2 + i b_1 C_2 S_1 & \\ \times (C_3 - i b_3 S_3) & +i (b_2 S_3 - b_1 C_3) S_1 & -b_2 C_1 S_2 - i b_1 C_2 S_1 & 1 - b_3 S_1 S_2 + C_1 C_2 \end{pmatrix}. \quad (19)$$

Note how the dynamical map carries the influence of ρ^B through the parameters b_1 , b_2 , b_3 . Thus, given any known initial state ρ^A and sufficiently detailed evolution $\rho^A(t)$, its interaction with another unknown ρ^B can be reconstructed, and even used to determine the parameters for the unknown state.

The most general map on a qubit can be implemented as the contraction of the unitary evolution of the given qubit coupled to at most two other qubits. We have restricted to the case where there is only one other qubit, thereby excluding certain maps. If ρ^B is allowed to be a mixed state, the family of dynamical maps we are excluding by choosing a one qubit environment is very small [23].

Consider the case where $\rho^A(0)$ is pure while $\rho^B(0)$ is fully mixed. Assume some interaction where the only non-zero parameters are $a_1 = \gamma_1 = \gamma_2 = \gamma_3 = 1$. Figure 1 shows how the purity of $\rho^A(t)$ and the purity of $\rho^B(t)$ change with time. As time progresses, they start

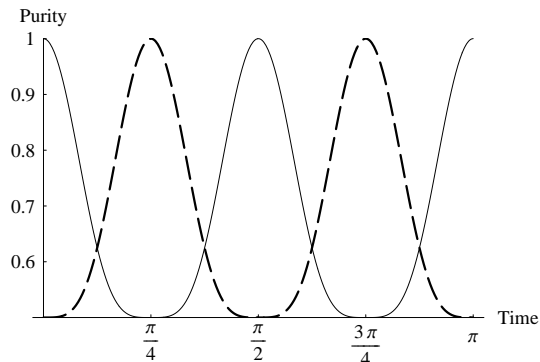


FIG. 1: Purity as a function of time for $a_1 = \gamma_2 = \gamma_3 = 1$, $a_2 = a_3 = b_1 = b_2 = b_3 = \gamma_1 = 0$. The solid line represents $P^A(t)$ while the dashed line represents $P^B(t)$. At $t = \pi/4$ purity has been totally swapped.

to get entangled and exchange purity through entanglement. $P^A(t)$ and $P^B(t)$ are equal at $t = \pi/8$, where some purity has been lost to entanglement. At $t = \pi/4$ they become separable again [24] purifying ρ^B at the expense of ρ^A , a dynamical process we call *purity swapping*.

We remark that for a weak coupling of this type, at

very short times, $P^A(\delta t)$ can only decrease, except in the case where no evolution happens. This can be considered to be a mechanism of decoherence. We can model a reservoir as a stream of $\{\rho_i^B\}$, where each of them interact independently for a short amount of time [25], swapping some purity from ρ^A to each ρ_i^B , but stopping the coupling before there is enough time to return the stolen purity. This corresponds to acting with the dynamical map from Eq. (19) in sequence:

$$\rho^A \rightarrow \mathfrak{B}^{(t_{n-1}, t_n)} \circ \mathfrak{B}^{(t_{n-2}, t_{n-1})} \circ \dots \circ \mathfrak{B}^{(t_0, t_1)} (\rho^A). \quad (20)$$

By controlling the strength, duration and number of these reservoir interactions it is possible to model decoherence processes, as well as desired, using only a finite number of degrees of freedom for the reservoir.

Another interesting regime to study is related to the creation of maximally entangled Bell states. Assume that we have initially two pure states, $\{a_1 = 1, a_2 = a_3 = 0\}$, $\{b_2 = 1, b_1 = b_3 = 0\}$. At a specific time t_{bell} , each of the qubits' purity goes to a minimum. For $\gamma_3 = 1$, the minimum is at $t_{bell} = \pi/4$, and we get that $E^{AB}(t_{bell})$ is:

$$\frac{1}{4} [\mathbb{1}^A \otimes \mathbb{1}^B + \sigma_1^A \otimes \sigma_2^B - \sigma_2^A \otimes \sigma_3^B + \sigma_3^A \otimes \sigma_1^B], \quad (21)$$

that, given some freedom to choose the basis for ρ^B , would be equivalent to the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. A similar, experimentally feasible time-reversed procedure would be responsible for extracting purity out of entanglement [13].

How much can an entanglement creation experiment be improved? From Eq. (11) an entanglement optimization protocol can be determined, given certain reasonable constraints. Lets illustrate this with a simple example for the case of the creation of entanglement from initially pure qubits.

Imagine some experimental setup that prepares two qubits with the Bloch vectors \mathbf{a} and \mathbf{b} representing their states oriented at some angle with respect to each other. The aim is to entangle these two. Assume that only two kinds of couplings between them are allowed. We act first with one kind followed by the other. The only thing that can be controlled is the duration for which each coupling is used. The total time available is restricted by

decoherence. How long should we act with each of them to maximize the entanglement rate?

Let the two qubits ρ^A and ρ^B , initially be in the states given by $a_1 = 1$, $a_2 = a_3 = 0$ and $b_1 = b_2 = 1/\sqrt{2}$, $b_3 = 0$. The allowed interactions are $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$ for some time $[0, t']$ and followed by $\gamma_3 = 1$, $\gamma_1 = \gamma_2 = 0$ at t' for an interval $(t', \pi]$. Using Eq. (11) we can calculate the state $E^{AB}(t)$ of the system at t' . Using $E^{AB}(t')$ as the new initial condition in Eq. (14) and the new coupling, $\gamma_3 = 1$ we can compute the state of the two qubits during $(t', \pi]$. Eq. (16) gives us the reduced density matrix of one of the qubits as a function of time from which we can compute its purity at every time.

What is the time t' that gives us the maximum entanglement? This protocol is not dependent on a particular measure of entanglement, but we will choose for simplicity the entropy of entanglement [3], which is a good measure as long as $E^{AB}(t)$ remains pure. Since it is monotonically related to the linear entropy [22, 26], the entanglement measure \mathcal{E} can be also chosen:

$$\mathcal{E} \sim 1 - \text{Tr} [(\rho^A)^2] = 1 - \text{Tr} [(\rho^B)^2] = 1 - P. \quad (22)$$

Using Eqs. (17) and (18), we find that for our choices of the parameters:

$$\mathcal{E} \sim 1 - \frac{3 \cos(8t') + 29}{32}. \quad (23)$$

which is maximized at $t' = \pi/8$, $3\pi/8$, $5\pi/8$ or $7\pi/8$.

This method can be applied to any interaction of the form Eq. (7), allowing us to optimize a very general class of entanglement creation procedures. It is significantly different from the procedure proposed in [14], in that we do not need to assume full control over the local transformations on each qubit at all times.

The general scheme of this entanglement optimization protocol can be summarized as follows:

1. Choose the initial conditions for the system, and express them in the form of either Eq. (4) or Eq. (14).
2. Identify the coupling parameters and constraints for Eq. (7). Compute the evolution due to the unitary operator from Eq. (9) by using the transformations in Eq. (11) or Eqs. (12,13).
3. Choose the measure of entanglement of your preference [3, 4, 5, 6, 7].
4. Considering the experimental constraints, optimize with respect to the desired parameters.

In summary, we have explicitly calculated the evolution generated by a general interaction between two qubits, opening the doors for studying all sorts of entangling interactions and experimentally realizable universal quantum gates. Through examples, we illustrated

how purity and entanglement are interchangeable quantities. We studied dynamical purity swapping, and its connection to decoherence phenomena. We described a procedure to generate Bell States from pure states, and its converse, purification. Finally, we proposed a practical protocol to optimize a given entangling procedure.

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* email: carod@physics.utexas.edu

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