Conformal Field Theory with Boundaries: The Cardy Conditions

by

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Abstract

Conformal field theory with boundaries are considered. The boundary conditions are shown to restrict the spectrum of primary fields allowed by the theory. The concept of boundary operators is introduced. The Cardy conditions that connect the boundary condtions to the fusion coefficients $\mathcal{N}^i_{jk}$ is derived. The conditions are applied to the Ising model as an example.

Contents

1 Introduction

The theory of conformal quantum fields (CFT) with boundaries has seen a large number of applications on simple models in statistical mechanics and in string theory. Since its first development, it has been clear that certain CFT models describe statistical systems at criticality with correlation functions of some operators being interpreted as order parameters. This correspondence has been done in the limit of infinitely large size of the system but in reality most of the systems of interest have finite physical dimensions. The finiteness of most physically interesting systems makes the study of the same CFT models in the presence of limiting boundaries important. We can then look how the system gets modified by the boundaries. Conformal field theories with boundaries has also seen its first application in open string theories in a closed string background. Later on, the same formalism was found to be important in the study of D-branes and their interactions.

In this paper we follow a very simple and pedagogical approach stressing the ideas behind the formalism that will be introduced. A basic knowledge of conformal field theories and modular invariance is assumed. In section 1.1 we give a general overview of physical situations with boundaries that are of interest in the contest of CFT. In particular we use D-branes and a magnetic system (Ising model) as guiding examples for this paper hoping this choice will avoid the risk of being too abstract. A brief description of the final result we reach is also given in this section. Section 2 presents a review of fusion rules, transformation of characters under modular invariance and the significance of Verlinde formula.

In section 3 conformal field theories with boundaries are introduced. It is shown first why in a general CFT there are two independent sets of generators of the Virasoro algebra. Conformal field theory on the upper half plane (UHP) is then introduced which ends up having only one set of generators. At this point we describe the method of images and finally we explain how is possible to have a CFT on an infinite strip. In section 4 we show how a conformal field theory defined on a system with boundaries can be described in two equivalent ways. For the two equivalent descriptions to be consistent with each other and with the boundary conditions we end up having to impose a set of restrictions on the action of Virasoro generators on boundary states. These conditions restrict the possible form these states can take leading to a solution found by Ishibashi. Under certain restrictions on the type of conformal field theory that we are considering, these states permits to derive a set of conditions named after Cardy linking the number of Verma modules present in the spectrum of the theory with the fusion numbers in the Verlinde formula.

1.1 Physical systems with boundaries

Any real physical system has a certain limited extension in space. This implies the existence of a limiting surface that confines a generic system to a specific volume of space. Clearly such a volume in general doesn't have to be finite in all directions. In our definition we include systems that are bounded along some directions and not along others (like a plane in $\mathbb{R}^3$). On the other hand not all the finite size systems have a limiting surface. We are therefore interested in those systems that, finite or infinite in space, allows some kind of geometrical object that can distinguish between what is the "system" and what is not. We will refer, from now on, to such an object as boundary. It is clear now that if our generic system can assume a certain number of microscopic configurations within the boundary, this may not be the case on the boundary itself. This is due to the fact that the boundary, in general, feels the influence of the inside and the outside of the system at the same time. So there may be some restrictions on the possible microscopic configurations of the boundary for each allowed configuration of the system. Moreover the limitation on the possible kind of boundaries can have a sort of feedback effect limiting the system itself. Here we can look at the problem from the exact opposite point of view. We can fix a specific configuration for the boundary and see which kind of possible configurations this choice allows for the system. However we look at the problem it is evident that a physical system with boundaries can assume a reduced number of possible states compared to one without boundaries.

1.2 Ising Model

In this paper we want to analyze this problem focusing our attention on quantum mechanical systems that have a manifest conformal invariance. One of the simplest examples of such a system is the Ising model. This model is a simplified version of a magnetic system made of an ensemble of elementary spins with two allowed orientations: plus and minus. Out of these two microscopic states we can build several reasonable macroscopic conditions for the boundary. How many of these can we have? Let's imagine that a function ϕ parametrizes all the possible choices. Since we know that at criticality the Ising model is described by a rational CFT, the function ϕ has to be a scaling field. In other words invariance under conformal transformations requires ϕ to be homogeneous as well as invariant along the boundary. These characteristics implies that only two kind of conditions are permitted: fixed and free. The first condition means that on the boundary all the spins are in the same configuration (+ or−), the second refers to exact opposite case when the spins on the boundary assume random values. In terms of values of ϕ, these are

$$\phi\mid_B = 0 \quad and \quad \phi\mid_B = \infty \tag{1}$$

respectively for free and fixed boundary condition. Out of this qualitative picture we can reach some conclusions about a simple magnetic system that approaches its critical point. Having $\phi = 0$ on the boundary means that as we approach the critical point the system undergoes an ordinary transition with bulk and boundary following the same evolution. On the other end $\phi = \infty$ implies that the boundary orders before the bulk signalling an extraordinary transition. These two different behavior are a direct consequence of having a finite system with a specific limiting surface as opposed to an infinitely large system.

1.3 Open strings and D-Branes

Another quantum mechanical system of interest that can be studied using conformal field theories with boundaries is in String theory. D-branes are objects that naturally arise when considering open strings obeying Dirichlet boundary conditions as opposed to von Neumann boundary conditions. For a bosonic open string this is equivalent to saying that

$$X^i(\sigma,\tau)\mid_{\substack{\sigma=0\\ \sigma=\pi}} = 0 \qquad with\ i = 1, ...p \quad (p < d-1). \tag{2}$$

In the d-dimensional space-time these conditions define a $(d-p)$-dimensional surface to which the ends of the open string are constrained. It turns out that such objects are not just artifacts of a mathematical condition on open strings but physical soliton like objects that admits interactions among themselves. Such an interaction is mediated, at low energies, by open strings stretching between them. In particular these interactions are given by the evaluation of the open string one loop diagram also known as annulus graph. On the other hand this diagram can be seen as a tree level propagation of a closed string between

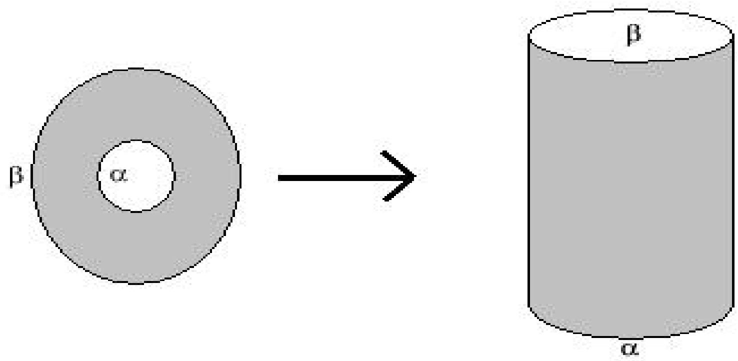

Figure 1: Annulus vs Cylinder

two D-branes. This implies that the closed string is created at one brane and destroyed at the second. So these two hypothetical D-branes can be viewed as the initial and final states of the propagation process of a closed string. As such they behave as boundaries for the tree level diagram that represents the closed string propagation. Since a physical process shouldn't depend on our description of it, it is clear that there should be a way to relate both pictures. This has to be done in a way that is consistent with the conformal invariance any string theory is believed to have. It is here that we find the crucial point this paper contains. If we abstract the picture from the specific physical setup that was used to create it, the underlying CFT is found to have a very specific structure. On the superficial level one fact becomes clear: since open strings have only one set of Virasoro generators while closed strings have two, consistency requires that closed strings with boundaries must have additional conditions originating from the boundaries themselves restricting the number of Virasoro generators that it can have. In short conformal field theories with boundaries can have only one set of Virasoro generators. In the course of finding the specific conditions to be imposed on generators to recover such a result we will extrapolate a much less obvious result. It will link the number of admissible primary fields $n^i_{\alpha\beta}$ appearing in the partition function of a CFT with boundaries

$$Z_{\alpha\beta} = \sum_i n^i_{\alpha\beta} \chi_i (q) \tag{3}$$

with the coefficients of the fusion rules (or fusion rules numbers) $\mathcal{N}^i_{kl}$ which appear in

$$\phi_k \times \phi_l = \sum_i \mathcal{N}^i_{kl} \phi_i \tag{4}$$

A key ingredient in getting to such a result will be the Verlinde formula.

2 Review: Fusion Rules and the Verlinde Formula

In the interest of making this presentation self contained we present in this section a review [1] of the fusion rules which describe the allowed results of the operator product expansions between members of the conformal families of a particular theory. We also show how the fusion coefficients $\mathcal{N}_{kl}^{i}$ are related to the $\mathcal{S}$ matrix that describes how the characters of a conformal field theory transform under modular transformations. We then proceed to show how $\mathcal{N}_{kl}^{i}$ are related to the boundary conditions imposed on a conformal field theory defined on a manifold with boundaries.

2.1 Verma Modules and Fusion Rules

Eigenstates of the generators of global conformal transformations L_0 and $\bar{L}_0$ (energy eignestates) of a conformal field theory can be expected to fall within representations of the local conformal algebra (Virasoro algebra). This is very much analogus to the way in which eigenstates of a rotationally invariant system fall into the irreducible representations of $SU(2)$.

In the theory of angular momentum we construct a set of energy states corresponding to a particular value of the total angular momentum **J** starting from a particular state $|j, m\rangle$ in which $j = m$. This state can be considered the "highest weight" eigenstate of the $SU(2)$ genereator J_z. Repeated applications of the lowering operator J_- on this highest weight state yields a set of states that span the representation space corresponding to that particular value of the total angular momentum.

2.1.1 Verma Modules

We can employ a similar strategy to obtain a set of energy eigenstates that span the representation space of the local conformal group. Such a set of states is called a Verma Module. No two generators in the Virasoro algebra

$$[L_n,\ L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{5}$$

commute with each other. So we choose a particular generator, L_0 which will be diagonal in the representation that we are going to construct. Let $|h\rangle$ denote a highest weight state of L_0 with eigenvalue h.

$$L_0|h\rangle = h|h\rangle$$

L_m $(m > 0)$ are the lowering operators for h and L_{-m} $(m < 0)$ are the raising operators. To ensure that the vacuum state $|0\rangle$ is invariant under all global conformal transformations we have to impose the condition that the highest weight state is annihilated by the lowering operators (this is unlike in the case of angular momentum). i.e.

$$L_n|h\rangle = 0 \qquad (n > 0)$$

A basis for the representation (all the descendant states belonging to the same family) is then built up by applying the raising operator in all possible ways on the highest weight state:

$$L_{-k_1}L_{-k_2}...L_{-k_n}|h\rangle \qquad (1 \le k_1 \le ... \le k_n)$$

[1] For a detailed treatment, see the term paper by Jason Hill and Sterling Garmon on *The Verlinde Formula and Rational Conformal Field Theories*

This state is then an eigenstate of L_0 with the eigenvalue

$$h' = h + k_1 + k_2 + ...k_n = h + N$$

where N is the level of the state.

A similar analysis can be done for the Verma modules associated with the antiholomorphic generators $\bar{L}_n$. If we denote by $V(c,h)$ and $\bar{V}(c,\bar{h})$ the Verma modules generated by the holomorphic and the antiholomorphic Virasoro generators for a value c of the central charge and conformal weights h and $\bar{h}$, the energy eigenstates belong to the tensor product $V \otimes \bar{V}$. The Hilbert space is in general a sum of such tensor products over all conformal dimensions of the theory:

$$\sum_{h,\bar{h}} V(c,h) \otimes \bar{V}(c,\bar{h})$$

Note that there may be several terms in the sum with the same conformal dimension

Any such representation of the Virasoro algebra constructed from a highest weight state $|h\rangle$ for a theory with central charge c is unitary if it does not contain negative norm states. Since the norm of the highest weight state and its descendants depends on h and c, unitarity imposes further restrictions on the values they can take.

2.1.2 Virasoro Characters

To a Verma module $V(c,h)$ generated by L_{-n} acting on the highest weight state $|h\rangle$, we can associate a generating function $\chi_{(c,h)}(\tau)$ called the character of the module. It is defined as

$$\chi_{(c,h)}(\tau) = \text{Tr } q^{L_0 - c/24} \tag{6}$$

where

$$q \equiv e^{2\pi i \tau}$$

A generic Virasoro character can be written in terms of the Dedekind η function as

$$\chi_{(c,h)}(\tau) = \frac{q^{h+(1-c)/24}}{\eta(\tau)}$$

The characters will be important in the discussion that follows because the $\mathcal{S}$ matrix that appears in the Verlinde formula describes how the characters transform under modular transformations rather than describing how the states themselves transform.

The Verma modules belonging to the antiholomorphic part of the theory also has analogous characters defined for them. In general the partition function of the theory can then be written as a sesquilinear combination of characters of the holomorphic and antiholomorphic characters. A character is in fact the partition function of the Verma module it corresponds to.

2.1.3 Fusion Rules

We already saw how energy eigenstates of a conformal field theory can be arranged into representations of the local conformal algebra called Verma modules. The highest weight state and its descendants in a

Verma module can be thought of as being created by the action of conformal fields on the vacuum. In particular the highest weight state is created by the action of a conformal primary field.

Now, going back to the analogy with the angular momentum algebra, we know that if we combine two states belonging to two different representations of the $SU(2)$ algebra, the combined state can be expressed as sum of states beloinging to various representations of the same algebra. i.e.

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{C}^{j}_{j_1 j_2} |j, m_1 + m_2\rangle$$

In the same manner we expect that the small distance product of two primary fields for a specific conformal field theory can be expressed as a sum of primary and descendant fields belonging to specific conformal families. The fusion rules, mentioned earlier in the introduction encode these model specific constraints on the operator products. As seen before, a generic fusion algebra may be written in the form

$$\phi_k \times \phi_l = \sum_k \mathcal{N}^i_{kl} \phi_i \tag{7}$$

$\mathcal{N}^i_{kl}$ are the fusion numbers which counts the multiplicity of paths from ϕ_k with ϕ_l to ϕ_i.

For a general conformal field theory with a generic value of the central charge c, there can be an infinite number of conformal families. For such a theory the fusion rules are of not much use. For certain values of c we can identify conformal field theories with finite set of conformal families, which closes under fusion. Such theories are called rational conformal field theories. We will be focusing mostly on this type of theories in the remaining discussion.

2.2 Modular Transformations and the $\mathcal{S}$ Matrix

In this section we adopt a different approach to study conformal field theories and look at the effect of modular transformations on the Virasoro characters of the theory. We see that for certain categories of conformal field theories, the transformation properties of the Virasoro characters can be expressed as a linear combination of the characters of a finite number of conformal families contained in the theory. This leads to the question whether the fusion rules of the theory can be related to the transformation properties of the characters under modular transformations.

2.2.1 Modular invariance

For reasons beyond the scope of the current article we are interested in studying conformal field theories defined on a torus with the additional restriction that the partition function of the theory be invariant under modular transformations. A torus constructed from a two dimensional plane is defined by two periods, ω_1 and ω_2. For any conformal field theory defined on such a torus, conformal invariance implies that the only relevant quantity is the modular parameter τ defined as $\tau = \omega_1/\omega_2$.

A modular transformation is defined as

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

with $a, b, c, d \in \mathcal{Z}$ and $ad - bc = 1$. Under such a transformation the modular parameter transforms as

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}$$

The generators of modular transformations can be identified as

$$\mathcal{T} : \tau \longrightarrow \tau + 1 \qquad \text{or} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{S} : \tau \longrightarrow -\frac{1}{\tau} \qquad \text{or} \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Under the action of the generators of modular transformations it can be shown that the Virasoro characters of a conformal field theory transform as follows

$$\mathcal{T} : \chi_i \longrightarrow e^{2\pi i h_i} \chi_i \tag{8}$$

$$\mathcal{S} : \chi_i \longrightarrow S_i^j \chi_j \tag{9}$$

Note that the modular transformation $\mathcal{S}$ mixes up the characters of different conformal families belonging to the theory just as the short distance product of two fields of different families yields a mixture of fields belonging to various families as given by the fusion rules of the theory. In fact it turns out that for minimal models and for rational confomal field theories we can find a relation between the modular transformation $\mathcal{S}$ of the characters and the fusion numbers $\mathcal{N}$. This relation is encapsulated in the Verlinde formula.

2.3 The Verlinde Formula

We present here the Verlinde formula in a general from without proof:

$$\sum_i S_i^j N_{kl}^i = \frac{S_k^j S_l^j}{S_0^j} \tag{10}$$

A remark is in order here about the relevance of this review section to the subject matter of the article. It will turn out that for rational conformal field theories defined on a manifold with boundaries, the number of copies $n_{\alpha\beta}^i$ of the representation of the Virasoro algebra labelled by i occuring in the spectrum when the boundary conditions are α and β is related to the fusion numbers $\mathcal{N}$. This relationship is established using the Verlinde formula that connects the fusion rules to modular transformations.

3 Conformal field theory with boundaries

In a generic conformal field theory the final target is to find the general form for correlation functions of primary fields ϕ_i. Under a conformal transformation $z \rightarrow w(z)$, $\overline{z} \rightarrow \overline{w}(\overline{z})$ these fields obey a simple scaling law

$$\phi_i(w, \overline{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\overline{h}} \phi_i(z, \overline{z}) \tag{11}$$

or in infinitesimal form

$$\delta_{\varepsilon,\overline{\varepsilon}}\phi_i = -\left(h\phi_i\partial_z\varepsilon + \varepsilon\partial_z\phi_i\right) - \left(\overline{h}\phi_i\partial_{\overline{z}}\overline{\varepsilon} + \overline{\varepsilon}\partial_{\overline{z}}\phi_i\right) \tag{12}$$

where $w = z + \varepsilon(z)$ and $\overline{w} = \overline{z} + \overline{\varepsilon}(\overline{z})$. These relations are a specific application of a more generic formula known as the Ward identity. From this identity it is possible to extract a series of generators that give rise to a specific algebra. The main feature of these generators is that we have not one but two sets of those satisfying the same algebra. To see why this is crucial let's look at them in more detail.

3.1 Ward identities and conformal generators

It is well known that in any reasonable field theory it is possible to define a energy-momentum tensor. This tensor plays a crucial role in identifying the conserved quantities (charges and currents) in the field theory. In a quantum field theory these conservation laws are codified in a series of relations involving correlation functions of operators (fields). If we restrict ourselves to a 2-dimensional space and we call X a string of n primary fields $\phi_1(x_1)$.......$\phi_n(x_n)$ these relations can be encapsulated in a general expression

$$\delta_\varepsilon \langle X \rangle = \int_M d^2x \partial_\mu \langle T^{\mu\nu}(x) \varepsilon_\nu(x) X \rangle \tag{13}$$

where $\varepsilon_\nu(x)$ encodes a generic infinitesimal transformation and the domain M contains all the positions of the fields in X. An infinitesimal conformal transformation is constrained by the equation

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = (\partial \cdot \varepsilon)\, \delta_{\mu\nu}. \tag{14}$$

These equations are nothing but the Cauchy-Riemann relations

$$\partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1, \qquad \partial_1 \varepsilon_1 = \partial_2 \varepsilon_2.$$

It is then natural to write $\varepsilon(z) = \varepsilon_1 + i\varepsilon_2$ and $\overline{\varepsilon}(\overline{z}) = \varepsilon_1 - i\varepsilon_2$ translating then from x_1, x_2 coordinates to $z, \overline{z} = x_1 \pm ix_2$ coordinates on the complex plane. The Ward identity transforms into the familiar form

$$\delta_{\varepsilon,\overline{\varepsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz\, \varepsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_C d\overline{z}\, \overline{\varepsilon}(\overline{z}) \langle \overline{T}(\overline{z}) X \rangle \tag{15}$$

where Gauss' theorem has been applied. Notice that z and $\overline{z}$ weren't independent variables, one being the conjugate of the other. The way the identity is written actually suggests the contrary. It seems, in fact, that we can consider independent infinitesimal variations of X on the complex plane for $\varepsilon(z)$ and $\overline{\varepsilon}(\overline{z})$. This strongly hints that we can actually consider z and $\overline{z}$ as independent variables. This is the power of conformal invariance and Ward identities! From this result we can built two independent sets of conformal generators defined in the standard way

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \qquad \overline{L}_n = \oint \frac{d\overline{z}}{2\pi i} \overline{z}^{n+1} \overline{T}(\overline{z}). \tag{16}$$

It is crucial to notice that this was possible because we are working with the infinite complex plane. These conclusions are not valid any more in the context of a bounded complex domain, as we will see in the next section.

3.2 Conformal generators in the presence of boundaries

In order to understand what happens to the previous statement regarding conformal generators in the presence of a boundary, let's consider a specific configuration: namely the upper half plane (UHP). The UHP is defined as $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and in the following we will be able to see that actually this is the prototype geometry of any complex domain with boundaries. Let us assume we have a specific condition on its boundary (the real axis) that we label by (α). Since we want to keep conformal invariance (both local and global), under an infinitesimal transformation the geometry of the UHP must be preserved. In particular this implies that $\varepsilon(z)$ must be real when z lies on the real axis. Same thing can be said for $\overline{\varepsilon}(\overline{z})$. So we reach the conclusion that if conformal invariance is a good symmetry of our system the boundary has to be mapped into itself by any conformal transformation. This requirement imposes a strong limitation to the admissible generators of the Virasoro algebra. We can easily deduce this consequence by writing down an analogous expression of the conformal Ward identity expressing the global variation of the action

$$-\frac{1}{2\pi i}\oint_{C^+} dz\;\varepsilon(z)\,T(z)+\frac{1}{2\pi i}\oint_{C^+} d\overline{z}\;\overline{\varepsilon}(\overline{z})\,\overline{T}(\overline{z})\,. \tag{17}$$

In this expression C^+ has to be a sufficiently large contour ultimately identified with a very large semicircle in the UHP. It is clear that the base of C^+ has to be the real axis, so that for this expression to vanish we have to set the condition $T=\overline{T}$ for $z=\overline{z}$. This is a necessary condition in order to preserve the invariance of the boundary condition (α). If this condition is valid for the variation of the action, the expression (17) has to be valid in general for any conformal Ward identity. Expressing this condition in terms of Cartesian coordinate may be illuminating. Recall that

$$\begin{aligned} T(z) &= \frac{1}{4}\left(T_{11}-2iT_{12}-T_{22}\right) \\ \overline{T}(\overline{z}) &= \frac{1}{4}\left(T_{11}+2iT_{12}-T_{22}\right) \end{aligned} \tag{18}$$

so that the condition $T=\overline{T}$ implies $T_{12}=0$ on the real axis. In field theory this is equivalent to saing that "no momentum flows across the boundary" We can now develop the theory in analogy to the complex plane and define the set of conformal generators to be

$$L_n=\oint_{C^+}\frac{dz}{2\pi i}z^{n+1}T(z)-\oint_{C^+}\frac{d\overline{z}}{2\pi i}\overline{z}^{n+1}\overline{T}(\overline{z})\,. \tag{19}$$

Thus only one set of generators survives. The geometry of the UHP and the condition imposed on the boundary suggests a different implementation of this definition. In fact since T and $\overline{T}$ are identified on the real axis we can define $\overline{T}(\overline{z})$ in the UHP as the analytic continuation of $T(z)$ in the lower half plane and redefine L_n as

$$L_n=\oint_{C^+}\frac{dz}{2\pi i}z^{n+1}T(z)+\oint_{C^-}\frac{dz}{2\pi i}z^{n+1}T(z)=\oint_{C}\frac{dz}{2\pi i}z^{n+1}T(z) \tag{20}$$

where now C is a full circle. A similar generalization can be done for the conformal Ward identity.

3.3 Method of images

The previous condition imposed on the holomorphic and antiholomorphic part of the energy-momentum tensor on the real axis suggests the extension of the conformal Ward identity to the whole plane. In order to do so we have to analytically continue the antiholomorphic part of the primary fields $\phi_i(z,\overline{z})$ into the lower half plane and perform the identification in perfect analogy with what was done for the tensor $\overline{T}$. So in a generic correlator we should regard the dependence of the correlator on antiholomorphic coordinates $\overline{z}_i$ on the UHP as a dependence on holomorphic coordinates z_i^* on the lower half plane. This is equivalent to setting up a mirror image of our domain in the lower half plane via a parity transformation. Notice the analogy of this technique with the one used in electrostatics in the presence of conducting surfaces. For this reason this method is called "method of images".

At this point we can build a generic product of fields X through the following simple identification

$$X=\phi_1(z_1,\overline{z}_1)\ldots\phi_n(z_n,\overline{z}_n)\sim\widetilde{X}=\phi_1(z_1)\,\overline{\phi}_1(z_1^*)\ldots\phi_n(z_n)\,\overline{\phi}_n(z_n^*) \tag{21}$$

where $\phi_i(z)$ stands for the holomorphic part of $\phi_i(z_i,\overline{z}_i)$ and $\overline{\phi}_i(z_i^*)$ for its antiholomorphic part. With such an identification we can write a compatible extension of the conformal Ward identity to the whole complex plane

$$\delta_\varepsilon \langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\, \varepsilon(z)\left\langle T(z)\, \widetilde{X}\right\rangle. \tag{22}$$

It has to be noted that the correlator $\langle X\rangle$ on the UHP satisfies the same differential equation (coming from conformal invariance) as that of the correlator $\langle \widetilde{X}\rangle$ on the entire plane regarded as a function of the $2n$ holomorphic variables z_i where $z_{n+i}=z_i^*$. So we have effectively replaced antiholomorphic degrees of freedom on the half plane, with holomorphic ones on the entire plane. The power of such an approach resides in the easiness with which the interaction of some field with the boundary is treated. For every boundary condition the interaction with local fields in the bulk is simulated via the interaction with their mirror images in the lower half plane. This has a remarkable consequence. We expect that the correlation function between two operators in the UHP to fall off has their separation increases along the x direction (real axis). This will remain in general true when such fields remain far from the boundary. The situation changes when local operators, separated by large distances along the x direction approach the boundary together. At this point interaction with the boundary through their mirror images becomes relevant and singular behavior can be observed. This is consistent with examples of the behaviour of magnetic materials in which the boundaries get magnetized before the bulk on approaching criticality. One such example where our field ϕ is treated as an order parameter is the Ising model

3.3.1 Example: Ising model

A simple application of the method of images is when we want to calculate the spin-spin correlation function of the Ising model in the UHP.

$$G(y_1,y_2;|z_1-z_2|) = \langle \sigma(z_1,\overline{z}_1)\,\sigma(z_2,\overline{z}_2)\rangle = \langle \sigma(z_1)\,\sigma(z_2)\,\sigma(z_1^*)\,\sigma(z_2^*)\rangle \tag{23}$$

where y_1 and y_2 are the distance of fields from the real axis (the imaginary part of their coordinates). As we already stated this correlation function will satisfy the same differential equation of a correlator depending on four holomorphic variables

$$\left[x(1-x)\frac{d^2}{dx^2}+\left(\frac{1}{2}-x\right)\frac{d}{dx}+\frac{1}{16}\right]F(x)=0 \tag{24}$$

with

$$\langle \sigma(z_1)\,\sigma(z_2)\,\sigma(z_3)\,\sigma(z_4)\rangle = \left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{14}z_{34}}\right)^{\frac{1}{8}} F(x) \tag{25}$$

$z_{ij}=|z_i-z_j|$ and $x=\frac{z_{12}z_{34}}{z_{13}z_{24}}$. This is a well known special case of hypergeometric equation that has two distinct solutions

$$F_\pm(x)=\sqrt{1\pm\sqrt{1-x}}. \tag{26}$$

At this point the presence of a boundary in our domain enters actively in determining which combinations of such solutions are allowed by the conditions imposed on the real axis. As we have seen in the introduction, a disordered surface (free condition) has an identically zero order parameter on the boundary. This implies that the spin correlation function for fixed y_1,y_2 goes to zero as $|z_1-z_2|$ goes to infinity. This corresponds in our language to $x\to-\infty$ so that the right combination is

$$F(x)=F_+(x)-F_-(x)\,. \tag{27}$$

On the contrary, the fixed boundary condition, when the surface of our domain orders before the bulk, requires the correlation function approaches infinity in the same limit. The combination

$$F\left(x\right)=F_{+}\left(x\right)+F_{-}\left(x\right)\sim\frac{1}{\left(y_{1}y_{2}\right)^{1/8}} \tag{28}$$

ensures this behavior. In conclusion, using the method of images and imposing the boundary conditions on the correlation function so obtained has the net effect of giving us a set of solutions for a CFT defined on the UHP.

$$G(y_{1},y_{2};|z_{1}-z_{2}|)=\frac{1}{\left(y_{1}y_{2}\right)^{1/8}}\sqrt{\tau^{1/4}\mp\tau^{-1/4}}\quad;\quad\tau=\frac{|z_{1}-z_{2}|+(y_{1}+y_{2})}{|z_{1}-z_{2}|+(y_{1}-y_{2})} \tag{29}$$

3.4 The infinite strip

Until now the only domain of the complex plane with boundaries that we studied in detail was the UHP. What about other configurations that don't possess such a simple geometry? The key ingredient is that any other geometry we can imagine can be transformed to the UHP using an appropriate conformal map. Any map of this kind has to have the additional property that it maps the boundary of the generic domain to the real axis. We won't demonstrate that this is always possible but just state it. The immediate consequence is that the formalism applied for the UHP and the conclusions deduced there can be applied to any bounded domain of the complex plane. A second less obvious consequence will come from the fact that a generic boundary may satisfy different conditions at the same time. When mapped onto the real axis, this would introduce a sort of discontinuity that has to be interpreted in the appropriate way. For clarity, we restrict our treatment to a specific example: the infinite strip.

An infinitely long strip, whose width we take to be L, is related to the UHP by a conformal mapping $w=\frac{L}{\pi}\ln z$. Here the positive real axis is mapped onto the lower edge of the strip meanwhile the negative real axis onto the upper edge. Writing $w=t+i\sigma$, the generators of time translation, or Hamiltonian H, is given in terms of the scale transformation generator on the UHP as

$$H=\int_{0}^{\pi}T_{tt}\;d\sigma=L_{0}-\frac{c}{24} \tag{30}$$

where the last term comes from the Schwartian derivative. It is clear that if we want the strip to be exactly like the UHP we considered, both edges of the strip must support the same boundary condition (α). However the existence of the generators of the Virasoro algebra depends only on the fact that the boundary conditions has to be conformally invariant. In formal language this means that $T_{t\sigma}=0$ on $\sigma=0,L$. This condition doesn't prevent us to considering the case when the two edges support different conditions (α,β). We will call from now on the corresponding Hamiltonian $H_{\alpha\beta}$.

Since our theory is still conformally invariant the eigenstates of the Hamiltonian will fall into irreducible representation of the Virasoro algebra. This conclusion doesn't exhaust the description of our system. In case of boundary conditions (α,β) on the strip, the corresponding UHP has a discontinuity at $z=0$. A discontinuity at this point implies that in the analytically continued space $\mathbb{C}$, the derivative with respect to z is not well defined any more. In the language of Virasoro generators this corresponds to a vacuum state that is no longer annihilated by L_{-1} (remember that regularity at $z=0$ requires $L_n\left|0\right\rangle=0$ for $n\geq-1$). We may consider this state as equivalent to the action of a certain operator $\phi_{\alpha\beta}\left(0\right)$ acting on the true vacuum $\left|0\right\rangle$. The possible operators that can be inserted and their relative scaling dimensions $h_{\alpha\beta}$ (highest weight) are found through the use of modular invariance applied to the partition function of the model considered.

A couple of remarks are in order here: a primary operator on the boundary may have a different scaling dimension than its counterpart in the bulk. Consequently an operator which is primary with respect to the usual Virasoro algebra may have no primary counterpart in the set of boundary operators. This may be well understood by recalling how such an operator is defined in terms of the corresponding highest weight state $|\phi\rangle = \phi(0)|0\rangle$. In particular the surface scaling dimension of a boundary operator determines how a two point function decays as a function of distance along the boundary, as we have seen in the Ising model at criticality. This is one of the fundamental ideas of CFT with boundaries. Juxtaposition of different conformally invariant boundary conditions is equivalent to the insertion of boundary operators on its singular points. This conclusion is true for the strip as well as for any kind of bounded domain in the complex plane.

Starting from the method of images we can find another interpretation for the boundary fields. We consider a bulk scaling field $\phi(z)$ on the upper half plane. As it approaches the real axis, this field can be thought of as interacting with the image field $\phi(z^*)$ and can be replaced by its OPE with its image:

$$\phi(z)\phi(z^*) \approx \sum_i (z - z^*)^{(h_i - 2h)} \phi_B^{(i)}(x)$$

where $x = (z + z^*)/2$ is the coordinate along the boundary. The fields $\phi_B^{(i)}$ are the boundary fields.

4 The Cardy Conditions

In the previous section we have encounterd the infinite strip which is obtained by applying a conformal transformation on the prototypical manifold with a boundary: the upper half plane. We proceed to analyse the infinite strip in more detail to arrive at the Cardy conditions.

4.1 Verlinde Formula from Boundary States

The basic strategy to arrive at a connection between boundary conditions and the Verlinde formula is to consider a conformal field theory defined on a finite cylinder quantized in two equivalent ways. In the first scheme, time flows around the cylinder. In this case the Hamiltonian which is the generator of time translations depend on the boundary conditions defined on the edges of the cylinder. This is very much like the propagation of an open string with specific boundary conditions at the ends of the string. The other quantization scheme is to let time flow along the length of the cylinder. In this case the Hamiltonian does not depend on the boundary conditions but rather propagates a particular boundary state defined on one edge of the cylinder to another boundary state defined on the other edge. This scheme is like the propagation of a closed string from one boundary state to another as mentioned in section 1.3.

4.1.1 Case 1: Time running around the cylinder

We construct the cylinder of finite length from the infinite strip by imposing periodicity along the length of the strip. Since time runs along the length of the strip, after imposing periodicity characterized by a period T, we will get a finite cylinder of circumference T with time running around it. The length of the cylinder will be L, which is the original width of the strip. Let boundary conditions labelled by α and β be imposed on the edges of the cylinder. Since the Hamiltonian in this case depends on the boundary conditions, we denote it by $(\pi T/L)H_{\alpha\beta}$. The prefactor is introduced to get the same normalization as L_0. The partition function for the theory on the cylinder can be written down as

$$Z_{\alpha\beta}(q) = \text{Tr } e^{-\left(\frac{\pi T}{L}\right)H_{\alpha\beta}} = \text{Tr } q^{H_{\alpha\beta}}$$

where $q \equiv e^{2\pi i\tau}$ and $\tau = iT/2L$.

If we assume local conformal invariance on the cylinder the spectrum of the Hamiltonian $H_{\alpha\beta}$ will arrange itself into irreducible representations of the Virasoro algebra (Verma modules). If $n^i_{\alpha\beta}$ is the number of copies of the representation labelled by i appearing in the spectrum, the partition function can be rewritten in terms of the characters of the corresponding Verma modules as shown below

$$Z_{\alpha\beta}(q) = \sum_i n^i_{\alpha\beta}\chi_i(q) \tag{31}$$

where the Virasoro character for a holomorphic Verma module is given in equation (6). Note that since for the finite cylinder we have only one set of Virasoro generators and therefore only one set of characters (holomorphic only), the partition function becomes a linear sum of characters instead of a sesquilinear sum.

in section 2 we saw that under a modular transformation of the form $\tau \longrightarrow -1/\tau$ the holomorphic characters transform as given in equation (9). Using this result we can rewrite the partition function as

$$Z_{\alpha\beta}(q) = \sum_{i,j} n^i_{\alpha\beta}\mathcal{S}_{ij}\chi_j(\tilde{q}) \qquad \tilde{q} \equiv e^{-2\pi i/\tau} \tag{32}$$

We have to keep in mind that the expressions for the partition function given in equations (31) and (32) are useful only if the sums on the right hand side contain a finite number of block diagonal terms. For this reason we are restricting ourselves to rational conformal field theories.

The reason for writing the partition function as given in (32) is actually to do a modular transformation on the cylinder which interchanges the roles of T and L. Under such a transformation the partition function becomes a trace over a Hamiltonian generating translations along the σ (space) direction. This leads to the second quantization scheme for the cylinder.

4.1.2 Case 2: Time running along the length of the cylinder

The transformation that we consider now maps the cylinder back onto the plane via the coordinate transformation

$$\zeta = e^{-2\pi i(t+i\sigma)/T} \qquad \text{or} \qquad w = i\frac{T}{2\pi}\ln\zeta$$

On the ζ plane the Virasoro generators are denoted by L^ζ_n and $\bar{L}^\zeta_n$ and the Hamiltonian that generates translations along the σ direction is

$$\hat{H} = \frac{2\pi}{T}\left(L^\zeta_0 + \bar{L}^\zeta_0 - \frac{c}{12}\right) \tag{33}$$

On the ζ-plane the boundaries of the cylinder are now concentric circles centered on the origin. On this annular region we can use radial quantization with time direction going radially outward from the center. This means that the boundary conditions are imposed by propagating the states from an initial state

$|\alpha\rangle$ residing on the inner boundary to a final state $|\beta\rangle$ residing on the outer boundary. The partition function is then expressed as

$$Z_{\alpha\beta}(q) = \langle\alpha|e^{L\hat{H}}|\beta\rangle = \langle\alpha|(\tilde{q}^{1/2})^{L_0^\zeta+\bar{L}_0^\zeta-c/12}|\beta\rangle \tag{34}$$

By doing the transformation to the ζ plane we have the advantage of already knowing the Hilbert space and also the fact that the holomorpic and antiholomorphic degrees of freedom propagate separately.

We still have to impose the condition that there cannot be any momentum transfer across the edges of the cylinder. This means that the holomorpic and antiholomorphic parts of the energy momentum tensor have to be equal to each other on the two boundaries. For the cylinder this condition is quite straight forward to write down:

$$T_{cyl}(0,t) = \bar{T}_{cyl}(0,t) \quad \text{and} \quad T_{cyl}(L,t) = \bar{T}_{cyl}(L,t)$$

if we map this condition on to the ζ plane we get the relation

$$T_{pl}(\zeta)\zeta^2 = \bar{T}_{pl}\bar{\zeta}^2 \qquad \zeta = e^{-2\pi it/T}.$$

In terms of the Virasoro generators acting on the boundary state $|\alpha\rangle$, this condition translates to

$$(L_n^\zeta - \bar{L}_{-n}^\zeta)|\alpha\rangle = 0 \tag{35}$$

A similar condition holds for $|\beta\rangle$ also. The conditions on the energy momentum tensor at the boundaries also make sure that the boundaries are invariant under conformal transformations. i.e. boundaries are mapped on to boundaries.

The states that satisfy the conditions given in equation (35) are called Ishibashi states. The variety of states that satisfies these conditions are very few. Such a state can be constructed as follows. Let $|j;N\rangle$ be a holomorphic state belonging to the Verma module j (N labels the level of the descendant states within that module) and $\overline{|j;N\rangle}$ be the corresponding antiholomorphic state. Now introduce a antiunitary operator U such that

$$U\overline{|j;0\rangle} = \overline{|j;0\rangle}^* \qquad U\bar{L}_n^\zeta = \bar{L}_n^\zeta U$$

Then the solution to equation (35) is a state of the form

$$|j\rangle \equiv \sum_N |j;N\rangle \otimes U\overline{|j;N\rangle} \tag{36}$$

To show that these states do indeed satisfy equation (35) we start from a generic state $\langle k;N_1|\otimes\overline{\langle l;N_2|}U^\dagger$ which form a basis to the dual space to vectors of the form $|j;N\rangle\otimes U\overline{|j';N'\rangle}$. Then

$$\begin{aligned}
\langle k;N_1|\otimes\overline{\langle l;N_2|}U^\dagger(L_n^\zeta - \bar{L}_{-n}^\zeta)|j\rangle &= \sum_N \langle k;N_1|L_n^\zeta|j;N\rangle\overline{\langle l;N_2|j;N\rangle}^* \\
&\quad - \sum_N \langle k;N_1|j;N\rangle\overline{\langle l;N_2|}U^\dagger\bar{L}_{-n}^\zeta U\overline{|j;N\rangle} \\
&= \delta_{kj}\delta_{il}(\langle j;N_1|L_n^\zeta|j;N_2\rangle - \langle j;N_2|\bar{L}_{-n}^\zeta|j;N_1\rangle^*) \\
&= 0
\end{aligned} \tag{37}$$

The boundary states $|\alpha\rangle$ and $|\beta\rangle$ will then be linear combinations of the states $|j\rangle$ belonging to different Verma modules. If the states $|j\rangle$ are normalized, the partition function can be written as

$$Z_{\alpha\beta}(q) = \sum_{i,j} \langle\alpha|i\rangle\langle i|(\tilde{q}^{1/2})^{L_0^\zeta+\bar{L}_0^\zeta-c/12}|j\rangle\langle j|\beta\rangle = \sum_j \langle\alpha|j\rangle\langle j|\beta\rangle\chi_j(\tilde{q}) \tag{38}$$

where

$$\langle i|(\tilde{q}^{1/2})^{L_0^\zeta+\bar{L}_0^\zeta-c/12}|j\rangle = \delta_{ij}\chi_i$$

Note that we have already made the identification that the Ishibashi states are constructed out of taking the linear combination of all the states in a particular Verma module with equal weights in writing equation (38). The issue of proving that the states $|j\rangle$ are unique and that it forms a complete basis for expanding the boundary states is also left untreated here. Comparing this result with the partition function for the case in which time ran around the cylinder given in equation (32) we make the identification

$$\sum_i \mathcal{S}_{ij} n^i_{\alpha\beta} = \langle\alpha|j\rangle\langle j|\beta\rangle \tag{39}$$

To proceed, we first identify the simplest kind of boundary state $|\tilde{0}\rangle$ such that the only representation occuring in the Hamiltonian $H_{\tilde{0}\tilde{0}}$ is the identity: $n^i_{\tilde{0}\tilde{0}} = \delta^i_0$. Such a state then satisfies the relation

$$\left|\langle\tilde{0}|j\rangle\right|^2 = \mathcal{S}_{0j}$$

In a unitary model the $\mathcal{S}$ matrix element $\mathcal{S}_{0j}$ can be shown to be positive and therefore the state $|\tilde{0}\rangle$ does have a positive norm and it does exist. This state can be taken as

$$|\tilde{0}\rangle = \sum_j \sqrt{\mathcal{S}_{0j}}|j\rangle$$

Similarly we can define a state

$$|\tilde{l}\rangle = \sum_j \frac{\mathcal{S}_{lj}}{\sqrt{\mathcal{S}_{0j}}}|j\rangle$$

This is a state such that $n^i_{\tilde{0}\tilde{l}} = \delta^i_l$. i.e. only the representation l propagates in $H_{\tilde{0}\tilde{l}}$. We can now rewrite equation (39) for the states of the type $|\tilde{l}\rangle$ rather than for states corresponding to arbitrary boundary conditions as follows

$$\sum_i \mathcal{S}_{ij} n^i_{\tilde{k}\tilde{l}} = \langle\tilde{k}|j\rangle\langle j|\tilde{l}\rangle = \frac{\mathcal{S}_{kj}\mathcal{S}_{lj}}{\mathcal{S}_{0j}} \tag{40}$$

This relation is identical to the Verlinde formula given in (10). We conclude from this that for rational conformal field theories defined on a manifold with boundaries, the number of each representation of the Virasoro algebra allowed in the spectrum by the boundary conditions is related to the fusion numbers of the theory by the relation

$$n^i_{\tilde{k}\tilde{l}} = \mathcal{N}^i_{kl} \tag{41}$$

This relation also helps us make sense of the notion of boundary operators introduced earlier. Assume that for a particular region on the upper edge of the infinite strip the bounday condition is given by $\tilde{0}$ and then it abruptly changes to $\tilde{k}$. Along the lower edge the boundary condition is always $\tilde{l}$ In the first region the Hamiltonian is $H_{\tilde{0}\tilde{l}}$ and states belonging to the l representation are allowed to propogate. When the boundary condition on the upper edge suddenly changes to $(\tilde{l}, \tilde{k})$, $\mathcal{N}^i_{kl}$ states of representation

i will propogate. This can be viewed as the action of the boundary operator $\phi_{\tilde{0}\tilde{k}}$. Since $\phi_{\tilde{0}\tilde{k}}$ transform under the representation k of the Virasoro algebra, we can see that the action of this operator on a representation transfoming as l should indeed yield $\mathcal{N}_{kl}^{i}$ states of representation i according to the usual fusion rules.

4.2 Application: The Ising Model

The modular $\mathcal{S}$ matrix for the Ising model is

$$\mathcal{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \sqrt{\frac{1}{2}} \\ \frac{1}{2} & \frac{1}{2} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \end{pmatrix}$$

where the three rows correspond to the representations with highest weights $h = 0$ (0), $h = \frac{1}{2}$ (ϵ) and $h = \frac{1}{16}$ (σ). The admissible boundary states are

$$\begin{aligned} |\tilde{0}\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle + \frac{1}{\sqrt{2}}|\sigma\rangle \\ |\tilde{\frac{1}{2}}\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle - \frac{1}{\sqrt{2}}|\sigma\rangle \\ |\tilde{\frac{1}{16}}\rangle &= |0\rangle - |\epsilon\rangle \end{aligned} \tag{42}$$

These boundary states are the realization on the ζ plane of a particular type of conformally invariant boundary condition. In the Ising model the three possible boundary conditions are to fix the spins at the boundary to be $+, -$ or let them be free. Since the first two states differ only by the sign of the odd operator σ we infer that these two correspond to the fixed boundary condtions.

Identifying the boundary operators that produce the transition between the boundary conditions is also straight forward. The operator ϕ_{+-} producing the transition from the $(+)$ to the $(-)$ boundary condition can be written as $\phi_{\tilde{0}\tilde{\frac{1}{2}}}$. Thus it transforms under the representation of weight $\frac{1}{2}$ of the Virasoro algebra. In the case of the Ising model the operator that scales in this fashion is $\phi_{(2,1)} = \phi_{(1,3)}$. Similarly the boundary operator ϕ_{+f} is identified with $\phi_{(1,2)} = \phi_{(2,2)}$. The results are summarized in the table below.

(α, β)	$h_{\alpha\beta}$
$(+,+)$ *or* $(-,-)$	0
(f,f)	$0, \frac{1}{2}$
$(+,-)$	$\frac{1}{2}$
$(+,f)$ *or* $(-,f)$	$\frac{1}{16}$

(43)

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