Quiz #1, Problem 1

(a) $$S_N = \sum_{n=0}^{N-1} R^n \cos n\theta$$

$$= \mathrm{Re}\Big(\underbrace{\sum_{n=0}^{N-1} (Re^{i\theta})^n}\Big)$$

$$= \frac{1 - R^N e^{iN\theta}}{1 - Re^{i\theta}}$$

This is the result for the sum of a geometric series:

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}$$

Now take the real part

$$= \mathrm{Re}\left[\frac{(1 - R^N e^{iN\theta})(1 - Re^{-i\theta})}{(1 - Re^{i\theta})(1 - Re^{-i\theta})}\right]$$

$$= \mathrm{Re}\left[\frac{1 - Re^{-i\theta} - R^N e^{iN\theta} + R^{N+1} e^{i(N-1)\theta}}{1 - 2R\cos\theta + R^2}\right]$$

$$\boxed{S_N = \frac{1 - R\cos\theta - R^N \cos N\theta + R^{N+1}\cos(N-1)\theta}{1 - 2R\cos\theta + R^2}}$$

Using $\cos(N-1)\theta = \cos\theta\cos N\theta + \sin\theta \sin N\theta$, one can also write this as

$$S_N = \frac{(1 - R\cos\theta)(1 - R^N \cos N\theta) + R^{N+1}\sin\theta \sin N\theta}{1 - 2R\cos\theta + R^2}$$

(b) $R = 1$ and $\theta = 2\pi m/N$, $m = 0, 1, \ldots, N$

The sum simplifies to $S_N = \mathrm{Re}\Big(\underbrace{\sum_{n=0}^{N-1} e^{i2\pi mn/N}}\Big)$

When $m=0$, the exponential is always 1, so $S_N = N$. When $m \neq 0$ we have a sum over the Nth roots of unity, which gives $S_N = 0$.

So $$\boxed{S_N = N\delta_{m0} = \begin{cases} N, & m=0 \\ 0, & m = 1, \ldots, N-1 \end{cases}}$$

The easiest way to get this from part (a) is to go back to

$$S_N = \mathrm{Re}\left(\frac{1-e^{iN\theta}}{1-e^{i\theta}}\right).$$

When $m=0$, it is easiest to take the limit $\theta \to 0$:

$$S_N = \mathrm{Re}\left(\frac{1-(1-iN\theta)}{1-(1-i\theta)}\right) = \mathrm{Re}\left(\frac{iN\theta}{i\theta}\right) = N$$

When $m = 1, \dots, d-1$, we have

$$S_N = \mathrm{Re}\left(\frac{1-e^{i2\pi m}}{1-e^{i2\pi m/N}}\right) = 0 \quad \text{because } e^{i2\pi m} = 1 \text{ and } e^{i2\pi m/N} \neq 1.$$

(c) $|R| < 1$, $N \to \infty \Rightarrow R^N \xrightarrow[N\to\infty]{} 0$, so

$$\boxed{\lim_{N\to\infty} S_N = \mathrm{Re}\left(\frac{1}{1-Re^{i\theta}}\right) = \frac{1-R\cos\theta}{1-2R\cos\theta+R^2}}$$