1.5. Newton's equation: $m\frac{d\vec{v}}{dt} = qB\hat{e}_z\times\vec{v}$

$$\underset{m\vec{a}}{\uparrow} \qquad \underset{\text{Lorentz force } \vec{F}}{\uparrow}$$

(a) $m\frac{d\vec{v}}{dt} = m\frac{dv_x}{dt}\hat{e}_x + m\frac{dv_y}{dt}\hat{e}_y$

$$qB\hat{e}_z\times\vec{v} = qB(v_x\underbrace{\hat{e}_z\times\hat{e}_x}_{\hat{e}_y} + v_y\underbrace{\hat{e}_z\times\hat{e}_y}_{-\hat{e}_x}) = -qBv_y\hat{e}_x + qBv_x\hat{e}_y$$

Now equate components: $m\frac{dv_x}{dt} = -qBv_y$ and $m\frac{dv_y}{dt} = +qBv_x$

Divide by m: $\boxed{\frac{dv_x}{dt} = -\omega v_y \text{ and } \frac{dv_y}{dt} = \omega v_x \qquad \omega = \frac{qB}{m}}$

(b) $z = v_x + iv_y$

$$\frac{dz}{dt} = \frac{dv_x}{dt} + i\frac{dv_z}{dt} = -\omega v_y + i\omega v_x = i\omega(v_x + iv_y) = i\omega z$$

$\Rightarrow \boxed{z(t) = z(0)e^{i\omega t}}$ ← rotation with angular frequency ω in v_x-v_y plane

(c) $v_x(t) + iv_y(t) = [v_x(0) + iv_y(0)][\cos\omega t + i\sin\omega t]$

$$= v_x(0)c - v_y(0)s + i(v_x(0)s + v_y(0)c)$$

$$\Rightarrow \boxed{\begin{aligned} v_x(t) &= v_x(0)\cos\omega t - v_y(0)\sin\omega t \\ v_y(t) &= v_x(0)\sin\omega t + v_y(0)\cos\omega t \end{aligned}}$$

(d) $w = x + iy$: $\frac{dw}{dt} = v_x + iv_y = z = z(0)e^{+i\omega t}$

Integrate: $w(t) = \underbrace{\left[w(0) - \frac{1}{i\omega}z(0)\right]}_{u} + \frac{1}{i\omega}z(0)e^{i\omega t}$

This solution describes circular motion about u.

$$\boxed{w(t) = u - \frac{i}{\omega}z(0)e^{i\omega t}}$$

$$\boxed{\begin{aligned} u &= w(0) + \frac{i}{\omega}z(0) \\ u_x &= x(0) - \frac{v_y}{\omega}, \quad u_y = y(0) + \frac{v_x}{\omega} \end{aligned}}$$

$$x(t)+iy(t) = u_x + iu_y - \frac{i}{\omega}\underbrace{\left(v_x(0)+iv_y(0)\right)\left(c+is\right)}_{v_x(0)c - v_y(0)s + i(v_x(0)s + v_y(0)c)}$$

$$= u_x + \frac{v_x(0)}{\omega}\sin\omega t + \frac{v_y(0)}{\omega}\cos\omega t$$

$$+ i\left[u_y - \frac{v_x(0)}{\omega}\cos\omega t + \frac{v_y(0)}{\omega}\sin\omega t\right]$$

$$\Longrightarrow \boxed{\begin{aligned} x(t) &= u_x + \frac{v_x(0)}{\omega}\sin\omega t + \frac{v_y(0)}{\omega}\cos\omega t \\ y(t) &= u_y - \frac{v_x(0)}{\omega}\cos\omega t + \frac{v_y(0)}{\omega}\sin\omega t \end{aligned}}$$

(e) Back to $\frac{d\vec{v}}{dt} = \omega\,\hat{e}_z\times\vec{v}$

Try $\vec{v}(t) = \vec{v}(0)\cos\omega t + \hat{e}_z\times\vec{v}(0)\sin\omega t$

$$\frac{d\vec{v}}{dt} = \omega\left[-\vec{v}(0)\sin\omega t + \hat{e}_z\times\vec{v}(0)\cos\omega t\right]$$

$$-\hat{e}_z\times\left[\hat{e}_z\times\vec{v}(0)\right] = -\hat{e}_z\left(\cancelto{0}{\hat{e}_z\cdot\vec{v}(0)}\right) + \vec{v}(0)\underbrace{\hat{e}_z\cdot\hat{e}_z}_{1} = \vec{v}(0)$$

BAC-CAB rule

or just think about the cross product

$$= \omega\,\hat{e}_z\times\left[\hat{e}_z\times\vec{v}(0)\sin\omega t + \vec{v}(0)\cos\omega t\right]$$

$$= \omega\,\hat{e}_z\times\vec{v}(t)$$ ✓ It works! This is a general way to describe circular motion

$$\boxed{\vec{v}(t) = \underbrace{\left(\cos\omega t\, I + \sin\omega t\,\hat{e}_z\times\right)}_{R(t)}\vec{v}(0) \Leftrightarrow \begin{pmatrix} v_x(t) \\ v_y(t)\end{pmatrix} = \begin{pmatrix}\cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix} v_x(0) \\ v_y(0)\end{pmatrix}}$$

(f) $\frac{d\vec{r}}{dt} = \vec{v}(t) \Longrightarrow$

$$\boxed{\begin{aligned}\vec{r}(t) &= \frac{1}{\omega}\left(\sin\omega t\, I - \cos\omega t\,\hat{e}_z\times\right)\vec{v}(0) \\ &= \frac{1}{\omega}\left(\sin\omega t\,\vec{v}(0) - \cos\omega t\,\hat{e}_z\times\vec{v}(0)\right)\end{aligned}}$$

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{\omega}\begin{pmatrix} \sin\omega t & \cos\omega t \\ -\cos\omega t & \sin\omega t \end{pmatrix}\begin{pmatrix} v_x(0) \\ v_y(0) \end{pmatrix}}$$

The case of interest for the drawing is $v_x(0) = 0$.