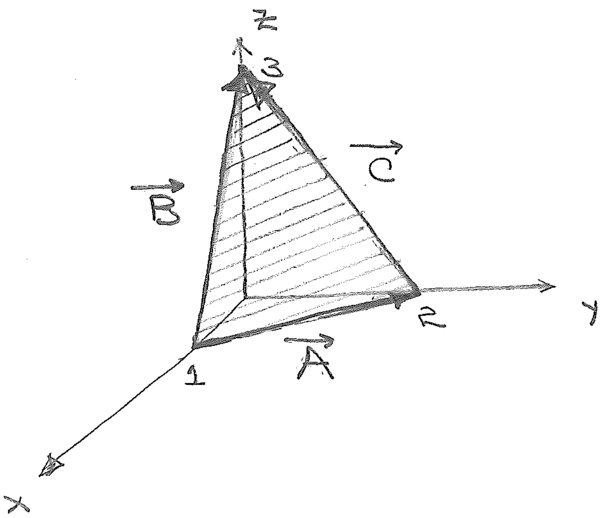
2.5



$$\vec{A} = 2\hat{e}_y - \hat{e}_x$$
$$\vec{B} = 3\hat{e}_z - \hat{e}_x$$
$$\vec{C} = 3\hat{e}_z - 2\hat{e}_y$$

(a)

① $\vec{A}$ and $\vec{B}$ define the shaded plane, so $\hat{n}$, the unit vector $\perp$ to the plane, is orthogonal to both $\vec{A}$ and $\vec{B}$. Thus

$$\boxed{\hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \frac{1}{7}\left(6\hat{e}_x + 3\hat{e}_y + 2\hat{e}_z\right)}$$

We use a + sign here so that we get the normal that points into the 1st quadrant

$$\vec{A}\times\vec{B} = (2\hat{e}_y - \hat{e}_x)\times(3\hat{e}_z - \hat{e}_x)$$

$$= 6\underbrace{\hat{e}_y\times\hat{e}_z}_{\hat{e}_x} - 2\underbrace{\hat{e}_y\times\hat{e}_x}_{-\hat{e}_z} - 3\underbrace{\hat{e}_x\times\hat{e}_z}_{-\hat{e}_y} + \underbrace{\hat{e}_x\times\hat{e}_x}_{0}$$

$$= 6\hat{e}_x + 3\hat{e}_y + 2\hat{e}_z$$

$$|\vec{A}\times\vec{B}| = (36+9+4)^{1/2} = 7$$

② We could also use $\vec{A}$ and $\vec{C}$

$$\hat{n} = \frac{\vec{A}\times\vec{C}}{|\vec{A}\times\vec{C}|} = \frac{1}{7}\left(6\hat{e}_x + 3\hat{e}_y + 2\hat{e}_z\right)$$

$$\vec{A}\times\vec{C} = (2\hat{e}_y - \hat{e}_x)\times(3\hat{e}_z - 2\hat{e}_y)$$

$$= 6\underbrace{\hat{e}_y\times\hat{e}_z}_{\hat{e}_x} - 4\underbrace{\hat{e}_y\times\hat{e}_y}_{0} - 3\underbrace{\hat{e}_x\times\hat{e}_z}_{-\hat{e}_y} + 2\underbrace{\hat{e}_x\times\hat{e}_y}_{\hat{e}_z}$$

$$= 6\hat{e}_x + 3\hat{e}_y + 2\hat{e}_z$$

$$|\vec{A}\times\vec{C}| = 7$$

Notice $\vec{C} = \vec{B} - \vec{A}$, so we actually know that

$$\vec{A}\times\vec{C} = \vec{A}\times\vec{B} - \underbrace{\vec{A}\times\vec{A}}_{0} = \vec{A}\times\vec{B}.$$

(3) We could also define a scalar field

$$f(x,y,z) = x + \frac{1}{2}y + \frac{1}{3}z$$

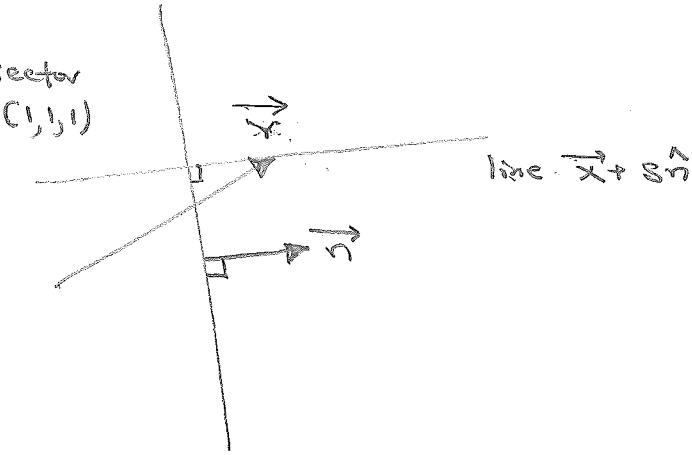
such that the plane is defined by $f = 1$.
Then

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{7}(6\hat{e}_x + 3\hat{e}_y + 2\hat{e}_z)$$

$$\nabla f = \hat{e}_x + \frac{1}{2}\hat{e}_y + \frac{1}{3}\hat{e}_z$$

$$|\nabla f| = \left(1 + \frac{1}{4} + \frac{1}{9}\right)^{1/2} = \frac{1}{6}\left(36 + 9 + 4\right)^{1/2} = \frac{7}{6}$$

(b) $\vec{r} = \hat{e}_x + \hat{e}_y + \hat{e}_z$ position vector of point (1,1,1)



1 The nearest point on the surface is found by moving $\perp$ to the surface, i.e., along the line $\vec{X} + s\hat{n}$, until one reaches the surface.

$$\text{line} \quad \vec{X} + \overbrace{s}^{\text{distance along } \hat{n}}\hat{n} = \underbrace{\left(1 + \tfrac{6}{7}s\right)}_{x}\hat{e}_x + \underbrace{\left(1 + \tfrac{3}{7}s\right)}_{y}\hat{e}_y + \underbrace{\left(1 + \tfrac{2}{7}s\right)}_{z}\hat{e}_z$$

The point on the line is in the surface when

$$1 = x + \tfrac{1}{2}y + \tfrac{1}{3}z = 1 + \tfrac{6}{7}s + \tfrac{1}{2} + \tfrac{3}{14}s + \tfrac{1}{3} + \tfrac{2}{21}s$$

$$= \underbrace{1 + \tfrac{1}{2} + \tfrac{1}{3}}_{1 + \frac{5}{6}} + \underbrace{\left(\tfrac{6}{7} + \tfrac{3}{14} + \tfrac{2}{21}\right)}_{\frac{36 + 9 + 4}{42} = \frac{49}{42}} s$$

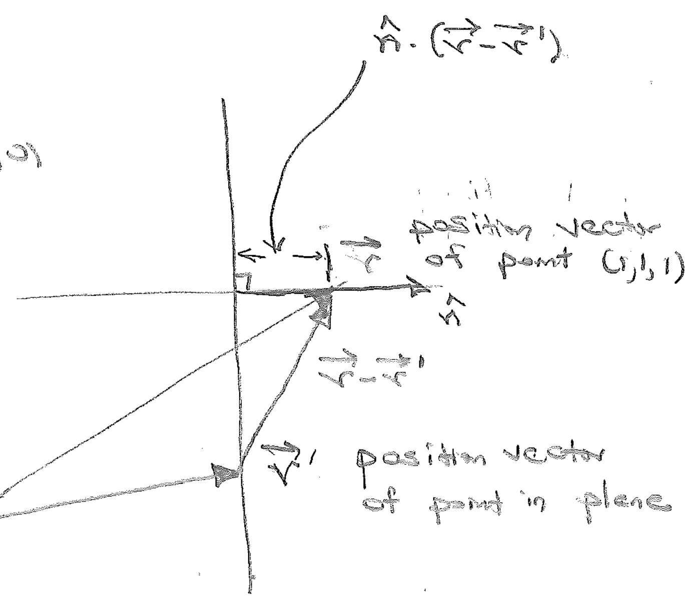
$$\Rightarrow \quad -\frac{5}{6} = \frac{49}{42}s \quad \rightarrow \quad s = -\frac{5}{\cancel{6}}\,\frac{\overset{7}{\cancel{42}}}{49} = -\frac{35}{49} = -\frac{5}{7}$$

$$\boxed{\begin{aligned} &\begin{pmatrix}\text{distance} \\ \text{to surface}\end{pmatrix} = |s| = \frac{35}{49} = \frac{5}{7} \\ &\begin{pmatrix}\text{nearest point} \\ \text{on surface}\end{pmatrix} = \vec{X} - \frac{5}{7}\hat{n} = \frac{19}{49}\hat{e}_x + \frac{34}{49}\hat{e}_y + \frac{39}{49}\hat{e}_y \\ &\qquad x = \frac{19}{49}, \quad y = \frac{34}{49}, \quad z = \frac{39}{49} \end{aligned}}$$

c

$\vec{r} = \hat{e}_x + \hat{e}_y + \hat{e}_z$

$\vec{r}\,' = \hat{e}_x \longleftarrow$ choose point (1,0,0) in plane



$\hat{n}\cdot(\vec{r}-\vec{r}\,')$ is the distance to the nearest point in the plane. The sign tells you which side of the plane (1,1,1) is on: positive means on the $+\hat{n}$ side; negative the $-\hat{n}$ side.

$$\vec{r}-\vec{r}\,' = \hat{e}_y + \hat{e}_z$$

$$\hat{n} = \frac{6}{7}\hat{e}_x + \frac{3}{7}\hat{e}_y + \frac{2}{7}\hat{e}_z$$

$$\hat{n}\cdot(\vec{r}-\vec{r}\,') = \frac{3}{7} + \frac{2}{7} = \frac{5}{7}$$

$$\boxed{\begin{pmatrix}\text{distance}\\ \text{to plane}\end{pmatrix} = S = \left|\hat{n}\cdot(\vec{r}-\vec{r}\,')\right| = \frac{5}{7}}$$

The nearest point has position vector

$$\vec{r} - \hat{n}S = \left(1-\frac{30}{49}\right)\hat{e}_x + \left(1-\frac{15}{49}\right)\hat{e}_y + \left(1-\frac{10}{49}\right)\hat{e}_z$$

$$\boxed{\vec{r}-\hat{n}S = \frac{19}{49}\hat{e}_x + \frac{34}{49}\hat{e}_y + \frac{39}{49}\hat{e}_z}$$