4.1

$$h(x,y) = 2xy - 3x^2 - 4y^2 - 18x + 28y + 12$$

$$\nabla h = \frac{\partial h}{\partial x}\hat{e}_x + \frac{\partial h}{\partial y}\hat{e}_y$$

$$= \underbrace{(2y - 6x - 18)}_{\partial h/\partial x}\hat{e}_x + \underbrace{(2x - 8y + 28)}_{\partial h/\partial y}\hat{e}_y$$

(a) The top of the hill is a maximum, where

$0 = \frac{\partial h}{\partial x} = \frac{\partial h}{\partial y}$, i.e., $0 = \nabla h$.

$$\boxed{\frac{\partial^2 h}{\partial x^2} = -6 \quad \frac{\partial^2 h}{\partial y^2} = -8 \quad \frac{\partial^2 h}{\partial x \partial y} = 2}$$

$$0 = \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} \Rightarrow 0 = y_0 - 3x_0 - 9$$

$$0 = \left.\frac{\partial h}{\partial y}\right|_{\substack{x=x_0\\y=y_0}} \Rightarrow 0 = x_0 - 4y_0 + 14$$

$y_0 = 3x_0 + 9$

$0 = x_0 - 4(3x_0 + 9) + 14$

$= -11x_0 - 22$

$x_0 = -2$, $y_0 = 3$

$\Rightarrow$ $x_0 = -2$ km

$y_0 = 3$ km

Top of hill is 2 km W and 3 km N of origin.

$$h(x_0,y_0) = \underbrace{2(-2)(3)}_{-12} - \underbrace{3(4)}_{-12} - \underbrace{4(9)}_{-36} - \underbrace{18(-2)}_{+36} + \underbrace{28(3)}_{84} + 12$$

$$\boxed{h(x_0,y_0) = 72 \text{ m}}$$

Notice that

$$h(x,y) = 72 - 3(x-x_0)^2 - 4(y-y_0)^2 + 2(x-x_0)(y-y_0)$$

We know this works because h is a quadratic function, specified by the position and height of the maximum and by the 2nd derivatives listed above, and this form gets all this right.

This form can be used to show that x_0, y_0 is actually a maximum (i.e., not a minimum or a saddle point):

$\Delta x \equiv x - x_0$
$\Delta y \equiv y - y_0$

$$-3(\Delta x)^2 - 4(\Delta y)^2 \le -3\left((\Delta x)^2 + (\Delta y)^2\right)$$
$$\le -\left((\Delta x)^2 + (\Delta y)^2\right)$$
$$\le -2(\Delta x)(\Delta y)$$

$(\Delta x + \Delta y)^2 \ge 0$

$\Leftrightarrow (\Delta x)^2 + (\Delta y)^2 \ge 2(\Delta x)(\Delta y)$

$-\left((\Delta x)^2 + (\Delta y)^2\right) \le -2(\Delta x)(\Delta y)$

$$\Downarrow$$
$$-3(\Delta x)^2 - 4(\Delta y)^2 + 2(\Delta x)(\Delta y) \le 0$$

$$\Longrightarrow \quad h(x,y) \le h(x_0, y_0) = 72$$

The official way to state this is that the matrix of 2nd derivatives has negative eigenvalues.

$$\begin{pmatrix} \partial^2 h/\partial x^2 & \partial^2 h/\partial x\partial y \\ \partial^2 h/\partial x \partial y & \partial^2 h/\partial y^2 \end{pmatrix} = 2\underbrace{\begin{pmatrix} -3 & 2 \\ 2 & -4 \end{pmatrix}}_{\text{eigenvalues } -7 \pm \sqrt{17}}$$

(b) $x_1 = 1, \; y_1 = 1$

$$\nabla h \Big|_{\substack{x = x_1 \\ y = y_1}} = \underbrace{(2y_1 - 6x_1 - 18)}_{-22} \hat{e}_x + \underbrace{(2x_1 - 8y_1 + 28)}_{22} \hat{e}_y$$

$$= 22(-\hat{e}_x + \hat{e}_y)$$

$$\boxed{|\nabla h| = 22\sqrt{1+1} = 22\sqrt{2} \approx 31.1 \text{ m/km} = 0.0311 \text{ m/m}}$$

∇h points in the direction $-\hat{e}_x + \hat{e}_y$, which is due northwest