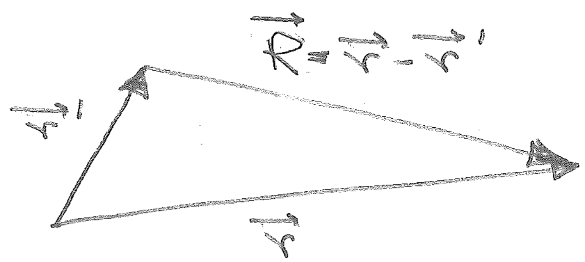
4.2

$$\vec{R} = \vec{r} - \vec{r}\,' = (x-x')\hat{e}_x + (y-y')\hat{e}_y + (z-z')\hat{e}_z$$



(a) $\nabla(R^2) = 2\vec{R}$

Method 1: Use the gradient in Cartesian coördinates:

$$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

$$\nabla(R^2) = \hat{e}_x \underbrace{\frac{\partial R^2}{\partial x}}_{= 2(x-x')} + \hat{e}_y \underbrace{\frac{\partial R^2}{\partial y}}_{2(y-y')} + \hat{e}_z \underbrace{\frac{\partial R^2}{\partial z}}_{2(z-z')}$$

So $\nabla(R^2) = 2\underbrace{\left[(x-x')\hat{e}_x + (y-y')\hat{e}_y + (z-z')\hat{e}_z\right]}_{\vec{R}}$

$$\boxed{\nabla(R^2) = 2\vec{R}}$$

Method 2: Put the origin of coördinates at the tip of $\vec{r}\,'$, so $\vec{r}\,' = 0$, and use spherical coördinates, so $\vec{r} = r\hat{e}_r$, $\vec{R} = \vec{r} = r\hat{e}_r$, and $R^2 = r^2$. Now use the gradient in spherical coördinates:

$$\nabla(R^2) = \nabla(r^2) = \underbrace{\frac{\partial r^2}{\partial r}}_{=2r}\hat{e}_r + \frac{1}{r}\underbrace{\frac{\partial r^2}{\partial \theta}}_{=0}\hat{e}_\theta + \frac{1}{r\sin\theta}\underbrace{\frac{\partial r^2}{\partial \phi}}_{=0}\hat{e}_\phi$$

$$= 2r\,\hat{e}_r$$

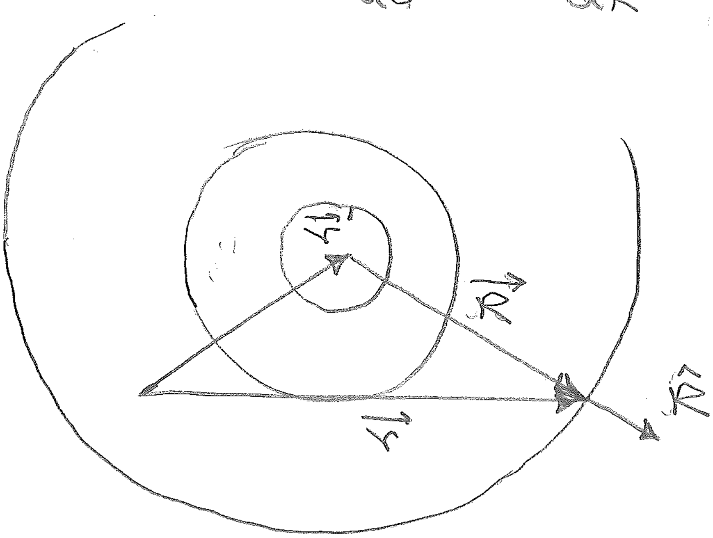
$$= 2\vec{R}$$

$$\therefore \boxed{\nabla(R^2) = 2\vec{R}}$$

(c) Method 3: Geometric approach: The level surfaces of R^2 are spheres centered at the tip of $\vec{r}'$! The unit vector normal to the level surface is $\hat{n} = \hat{R} = \dfrac{\vec{R}}{R}$, and $s = R$ measures length in the normal direction. The slope in the normal direction is

$$\frac{df}{ds} = \frac{dR^2}{dR} = 2R$$

$$\therefore \nabla(R^2) = \frac{dR^2}{ds}\hat{n} = \frac{dR^2}{dR}\hat{R} = 2R\hat{R} = \boxed{2\vec{R} = \nabla(R^2)}$$



This is really the same calculation as Method 2, but without knowing anything about spherical coördinates.

(b) $\nabla\left(\frac{1}{R}\right) = -\frac{\vec{R}}{R^3}$

Method 1:

$$\frac{1}{R} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}$$

$$\nabla\left(\frac{1}{R}\right) = \frac{\partial(1/R)}{\partial x}\hat{e}_x + \frac{\partial(1/R)}{\partial y}\hat{e}_y + \frac{\partial(1/R)}{\partial z}\hat{e}_z$$

$$= -\frac{1}{2}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-3/2} 2(x-x')$$

$$= -\frac{x-x'}{R^3}$$

$$= -\frac{y-y'}{R^3}$$

$$= -\frac{z-z'}{R^3}$$

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x-x')\hat{e}_x + (y-y')\hat{e}_y + (z-z')\hat{e}_z}{R^3}$$

$$= -\frac{\vec{R}}{R^3}$$

$$= -\frac{\hat{R}}{R^2}$$

$$\boxed{\nabla\left(\frac{1}{R}\right) = -\frac{\vec{R}}{R^3}}$$

Method 2: $\vec{r}\,' = 0$, $\vec{R} = \vec{r}$, $\hat{R} = \hat{e}_r$, $\frac{1}{R} = \frac{1}{r}$

$$\nabla\left(\frac{1}{R}\right) = \underbrace{\frac{\partial(1/r)}{\partial r}}_{-\frac{1}{r^2}}\,\hat{e}_r = \boxed{-\frac{\hat{R}}{R^2} = \nabla\left(\frac{1}{R}\right)}$$

Method 3:

$$\nabla\left(\frac{1}{R}\right) = \frac{d(1/R)}{ds}\,\hat{n} = \underbrace{\frac{d(1/R)}{dR}}_{-\frac{1}{R^2}}\,\hat{R} = \boxed{-\frac{\hat{R}}{R^2} = \nabla\left(\frac{1}{R}\right)}$$

Notice that Method 2 and Method 3 are essentially the same, because the spherical coördinates are matched to functions of R, i.e., $r = R$.

(c) Let's use Method 3:

$$\nabla(R^n) = \frac{dR^n}{dR}\,\hat{R} = \boxed{nR^{n-1}\,\hat{R} = nR^{n-2}\,\vec{R} = \nabla(R^n)}$$