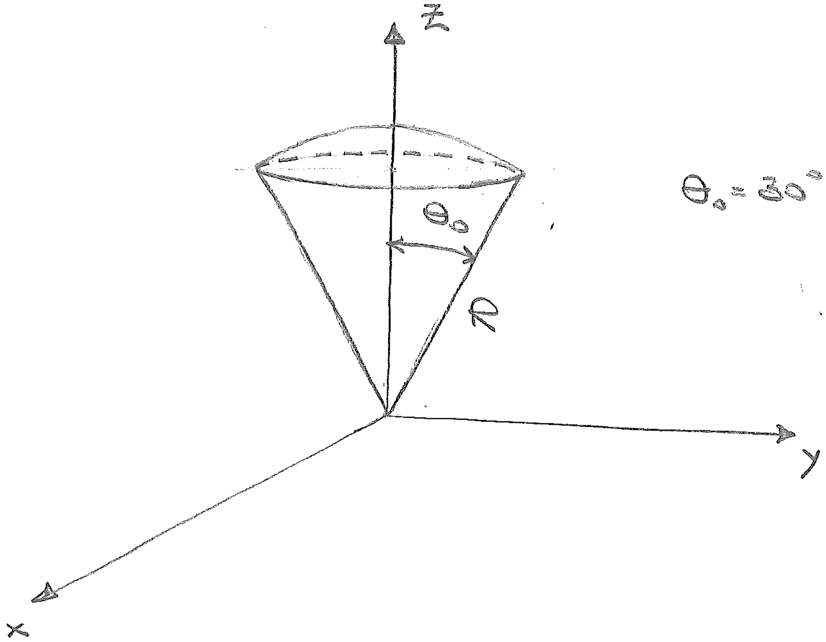

$\theta_o = 30^\circ = \frac{\pi}{6}$

The "ice-cream cone" volume is defined by

$$0 \le r \le R, \quad 0 \le \theta \le \theta_o = \frac{\pi}{6}, \quad 0 \le \phi \le 2\pi$$

The vector function is

$$\vec{V} = r^2 \sin\theta\,\hat{r} + 4r^2\cos\theta\,\hat{\theta} + r\tan\theta\,\hat{\phi}$$

$$\nabla\cdot\vec{V} = \frac{1}{r^2}\underbrace{\frac{\partial}{\partial r}(r^2 V_r)}_{\frac{\partial}{\partial r}(r^4 \sin\theta) \,=\, 4r^3\sin\theta} + \frac{1}{r\sin\theta}\underbrace{\frac{\partial}{\partial\theta}(\sin\theta\, V_\theta)}_{4r^2\frac{\partial}{\partial\theta}(\cos\theta\sin\theta)} + \frac{1}{r\sin\theta}\underbrace{\frac{\partial V_\phi}{\partial\phi}}_{=0}$$

$$= 4r\sin\theta + \frac{4r}{\sin\theta}\frac{\partial}{\partial\theta}(\cos\theta\sin\theta)$$

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$$\int_V \nabla\cdot\vec{v}\, d\tau = \int_0^R r^2 dr \int_0^{\theta_0} \sin\theta\, d\theta \int_0^{2\pi} d\phi \;\; \nabla\cdot\vec{v}$$

$d\tau$: $r^2 \sin\theta\, dr\, d\theta\, d\phi$

$\nabla\cdot\vec{v}$: no ϕ-dependence so the ϕ-integral gives 2π

$$= 8\pi \underbrace{\int_0^R r^3 dr}_{\frac{1}{4}R^4} \int_0^{\theta_0} \sin^2\theta\, d\theta$$

$$+ \, 8\pi \underbrace{\int_0^R r^3 dr}_{\frac{1}{4}R^4} \underbrace{\int_0^{\theta_0} d\theta \, \frac{\partial}{\partial\theta}(\sin\theta\cos\theta)}_{\sin\theta\cos\theta\big|_0^{\theta_0}}$$

$$\sin\theta\cos\theta\Big|_0^{\theta_0} = \sin\theta_0\cos\theta_0 = \frac{1}{2}\sin 2\theta_0$$

$$\int_0^{\theta_0} d\theta \sin^2\theta = \int_0^{\theta_0} d\theta \, \frac{1}{2}(1-\cos 2\theta) = \frac{1}{2}\left(\theta_0 - \frac{1}{2}\sin 2\theta_0\right)$$

$$\int_V \nabla\cdot\vec{v}\, d\tau = \cancel{8}\pi \, \frac{1}{\cancel{4}}R^4 \, \frac{1}{\cancel{2}}\left(\theta_0 - \frac{1}{2}\sin 2\theta_0\right)$$

$$+ \, \cancel{8}\pi \, \frac{1}{\cancel{4}} R^4 \frac{1}{\cancel{2}} \sin 2\theta_0$$

$$= \pi R^4\left(\theta_0 - \frac{1}{2}\sin 2\theta_0 + \sin 2\theta_0\right)$$

$$\boxed{\int_V \nabla\cdot\vec{v}\, d\tau = \pi R^4 \underbrace{\left(\theta_0 + \frac{1}{2}\sin 2\theta_0\right)}_{\theta_0 = \frac{\pi}{6} \to \; = \frac{\pi}{6} + \frac{1}{2}\sin 60^\circ = \frac{\pi}{6} + \frac{\sqrt{3}}{4}} = \frac{\pi R^4}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)}$$

2 $\displaystyle\int_V \nabla\cdot\vec{v}\,d\tau = \int_S \vec{v}\cdot\vec{da}$ (divergence theorem)

$$= \int_{\underbrace{top}_{\substack{r=R \\ 0\le\theta\le\theta_0 \\ 0\le\phi\le 2\pi}}} \vec{v}\cdot\underbrace{\vec{da}}_{\substack{\hat{r}\, R\,d\theta\, R\sin\theta\, d\phi \\ =\hat{r}\, R^2 \sin\theta\, d\theta\, d\phi}} + \int_{\underbrace{side}_{\substack{\theta=\theta_0 \\ 0\le r\le R \\ 0\le\phi\le 2\pi}}} \vec{v}\cdot\underbrace{\vec{da}}_{\substack{\hat{\theta}\, dr\; r\sin\theta_0\, d\phi \\ =\hat{\theta}\; r\sin\theta_0\, dr\, d\phi}}$$

$$\int_{top} \vec{v}\cdot\vec{da} = \int \underset{\substack{\| \\ R^2\sin\theta}}{v_r}\, R^2\sin\theta\, d\theta\, d\phi$$

$$= R^4 \underbrace{\int_0^{\theta_0} d\theta\, \sin^2\theta}_{\frac{1}{2}(\theta_0 - \frac{1}{2}\sin 2\theta_0)} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \pi R^4 (\theta_0 - \tfrac{1}{2}\sin 2\theta_0)$$

$$\int_{side} \vec{v}\cdot\vec{da} = \int \underset{\substack{\| \\ 4r^2\cos\theta_0}}{v_\theta}\, r\sin\theta_0\, dr\, d\phi$$

$$= 4\sin\theta_0\cos\theta_0 \underbrace{\int_0^R dr\, r^3}_{\frac{1}{4}R^4} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\int_{\text{side}} \vec{v} \cdot \vec{da} = 2\pi R^4 \sin\theta_0 \cos\theta_0 = \pi R^4 \sin 2\theta_0$$

$$\boxed{\begin{aligned} \int_S \vec{v} \cdot \vec{da} &= \pi R^4 \left(\theta_0 - \tfrac{1}{2}\sin 2\theta_0 + \sin 2\theta_0\right) \\ &= \pi R^4 \left(\theta_0 + \tfrac{1}{2}\sin 2\theta_0\right) \end{aligned}}$$