4.4, The divergence theorem and vector integrals

(a)

$$\nabla\cdot(f\vec{F}) = \sum_j \frac{\partial}{\partial x_j}(fF_j)$$

$$= \sum_j \frac{\partial f}{\partial x_j} F_j + \sum_j f \frac{\partial F_j}{\partial x_j}$$

$$= \nabla f \cdot \vec{F} + f\,\nabla\cdot\vec{F}$$

4.(b) $\hat{e}\cdot\int_V \nabla f \, d\tau = \int_V \hat{e}\cdot\nabla f \, d\tau$ ⟵ because $\hat{e}$ is constant

$= \int_V \nabla\cdot(f\hat{e})\,dv$ ⟵ because $\nabla\cdot(f\hat{e}) = \nabla f\cdot\hat{e} + f\nabla\cdot\hat{e} = \nabla f\cdot\hat{e}$ because $\nabla\cdot\hat{e} = 0$

$= \oint_S f\hat{e}\cdot\vec{da}$ ⟵ divergence theorem

$= \hat{e}\cdot\oint_S f\,\vec{da}$ ⟵ because $\hat{e}$ is constant

$$\therefore \boxed{\int_V \nabla f \, d\tau = \oint_S f\,\vec{da}}$$

because the two vectors have the same components in 3 orthogonal directions.

constants

(c) Let $f = \vec{A}\cdot\vec{r} = A_x x + A_y y + A_z z$

$$\nabla f = \underbrace{\frac{\partial f}{\partial x}}_{=\,A_x}\hat{e}_x + \underbrace{\frac{\partial f}{\partial y}}_{=\,A_y}\hat{e}_y + \underbrace{\frac{\partial f}{\partial z}}_{=\,A_z}\hat{e}_z$$

$$= A_x\hat{e}_x + A_y\hat{e}_y + A_z\hat{e}_z$$

$$\nabla f = \vec{A}$$

$$\oint_S \vec{A}\cdot\vec{r}\,\vec{da} = \int_V \vec{A}\,d\tau \qquad \longleftarrow \text{part (a)}$$

$$= \vec{A}\underbrace{\int_V d\tau}_{\frac{4}{3}\pi b^3}$$

$$\boxed{\oint_S \vec{A}\cdot\vec{r}\,\vec{da} = \vec{A}\,\frac{4}{3}\pi b^3}$$

Direct evaluation of integral surface integral

Orient coördinate axes so that $\vec{A}$ is in the z direction, i.e., $\vec{A} = A\hat{e}_z$.

$$\vec{A}\cdot\vec{r} = A\hat{e}_z\cdot\vec{r} = Az = Ar\cos\theta \overset{\text{on } S}{=} Ab\cos\theta$$

$$\vec{da} = \hat{e}_r b^2\sin\theta\, d\theta\, d\phi$$

$$\oint_S \vec{A}\cdot\vec{r}\,\vec{da} = \int A b \cos\theta\, \hat{e}_r\, b^2 \sin\theta\, d\theta\, d\phi$$

$$= A b^3 \int_0^{\pi} d\theta \cos\theta \sin\theta \int_0^{2\pi} d\phi\, \hat{e}_r$$

$$\int_0^{2\pi} d\phi\, \hat{e}_r = \int_0^{2\pi} d\phi \left(\hat{e}_z \cos\theta + \hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi\right)$$

The $\hat{e}_x$ and $\hat{e}_y$ components integrate to 0

$$= \int_0^{2\pi} d\phi\, \hat{e}_z \cos\theta$$

$$= 2\pi \cos\theta\, \hat{e}_z$$

$$\oint_S \vec{A}\cdot\vec{r}\,\vec{da} = 2\pi b^3 \underbrace{A\hat{e}_z}_{\vec{A}} \underbrace{\int_0^{\pi} d\theta \cos^2\theta \sin\theta}_{\int_{-1}^{+1} du\, u^2 = \frac{2}{3}}$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$\oint_S \vec{A}\cdot\vec{r}\,\vec{da} = \frac{4}{3}\pi b^3 \vec{A}$$