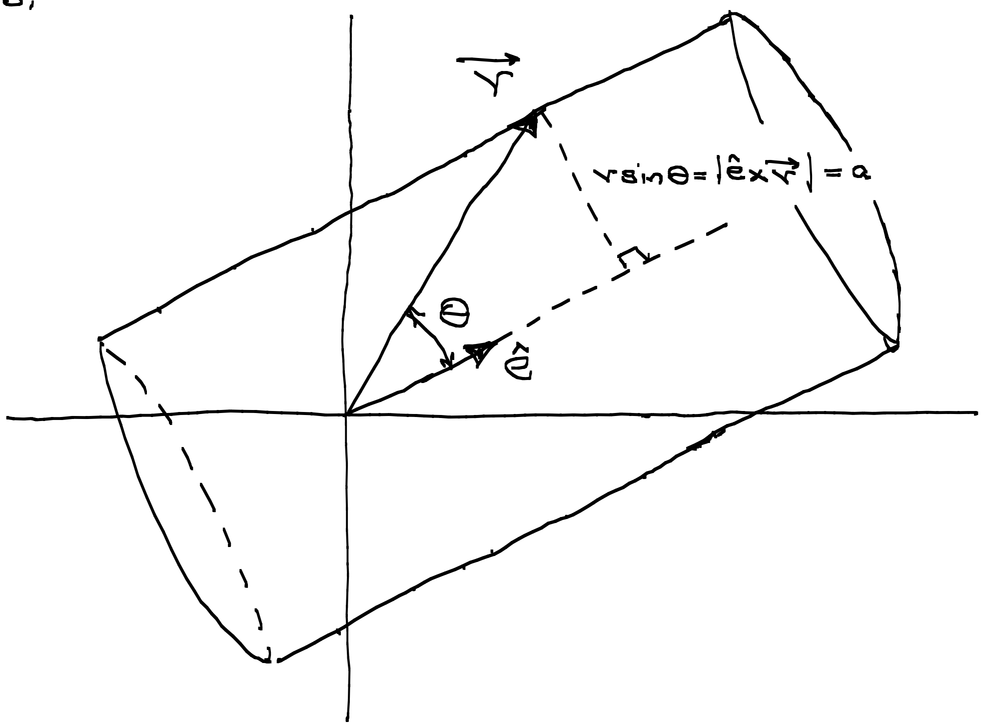
4.5.



(a) $x^2 + y^2 + z^2 - xy - yz - zx = 9$

This equation is invariant under a 120° rotation about the axis $\hat{e} = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})$, which takes $x \to y$, $y \to z$, $z \to x$, which means that $\hat{e}$ is the axis of symmetry. The point $\vec{r} = 3\hat{x}$ satisfies the equation, so the radius is

$$a = |\hat{e}\times\vec{r}| = \sqrt{3}\,|\hat{z} - \hat{y}| = \sqrt{6}.$$

One can solve for this directly by saying $\hat{e} = u\hat{x} + v\hat{y} + w\hat{z}$, with $u^2 + v^2 + w^2 = 1$. Then

$$a^2 = |\hat{e}\times\vec{r}|^2 = (vz - wy)^2 + (wx - uz)^2 + (uy - vx)^2$$

$$= x^2(w^2 + v^2) + y^2(w^2 + u^2) + z^2(v^2 + u^2)$$

$$- 2uvyx - 2vwyz - 2wuxz$$

$$\Rightarrow w^2 + v^2 = w^2 + u^2 = v^2 + u^2 \Rightarrow u^2 = v^2 = w^2 = \frac{1}{3}$$

$$2uv = 2vw = 2wu = \frac{2}{3} \Rightarrow u = v = w = \pm\frac{1}{\sqrt{3}}$$

This is the ambiguity in the direction of $\hat{e}$. Let's choose the + sign.

So we have $\hat{e} = \frac{1}{\sqrt{3}}(\hat{x}+\hat{y}+\hat{z})$ and

$$a^2 = \frac{2}{3}(x^2+y^2+z^2-xy-yz-xz) \Rightarrow \frac{3}{2}a^2 = 9 \Rightarrow a = \sqrt{6}$$

(b) The cylinder is a level surface of the function

$$f(x,y,z) = x^2+y^2+z^2-xy-yz-xz,$$

so we get the unit outward normal by taking the gradient and normalizing:

$$\frac{\partial f}{\partial x} = 2x-y-z, \quad \frac{\partial f}{\partial y} = 2y-x-z, \quad \frac{\partial f}{\partial z} = 2z-x-y$$

For the point $\vec{R} = 4\hat{x}+\hat{y}+\hat{z}$ (one should check that the tip of $\vec{R}$ is on the cylinder, but it is), we have

$$\nabla f = 6\hat{x} - 3\hat{y} - 3\hat{z} = 3(2\hat{x}-\hat{y}-\hat{z})$$

$$\hat{n} = \frac{2\hat{x}-\hat{y}-\hat{z}}{|2\hat{x}-\hat{y}-\hat{z}|} = \boxed{\frac{1}{\sqrt{6}}(2\hat{x}-\hat{y}-\hat{z}) = \hat{n}}$$

(c) $\cos\theta = \dfrac{\hat{n}\cdot\vec{R}}{|\vec{R}|} = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$$\hat{n}\cdot\vec{R} = \frac{1}{\sqrt{6}}(8-1-1) = \sqrt{6}$$

$$|\vec{R}| = \sqrt{18} = 3\sqrt{2}$$

(d) The tangent plane is the surface orthogonal to $\hat{n}$ such that the tip of $\vec{R}$ is in the plane, i.e.,

$$\hat{n}\cdot\vec{r} = \text{constant} = \hat{n}\cdot\vec{R} = a = \sqrt{6}$$

because $\vec{R}$ is a point in the plane

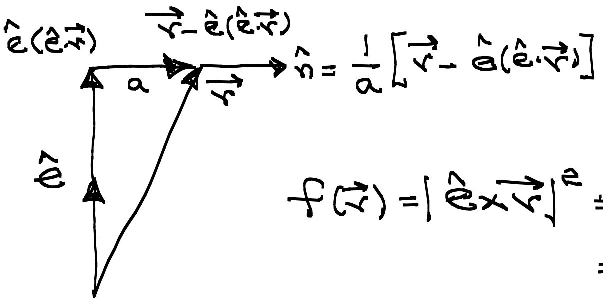
defines planes

$$\hat{n}\cdot\vec{r} = \frac{1}{\sqrt{6}}(2x-y-z)$$

So the equation of the tangent plane is

$$\boxed{2x-y-z = 6}$$

Actually, all this is easier if you work generally and and directly with the vector notation and a picture
The cylinder is a level surface of the function



$$f(\vec{r}) = |\hat{e}\times\vec{r}|^2 = \hat{e}\times\vec{r}\cdot\hat{e}\times\vec{r}$$

$$= \hat{e}\cdot\underbrace{\left[\vec{r}\times(\hat{e}\times\vec{r})\right]}_{\hat{e}(\vec{r}\cdot\vec{r})-\hat{r}(\vec{r}\cdot\hat{e})} \longleftarrow \text{cyclic property of the scalar triple product}$$

$\longleftarrow$ BAC-CAB rule

$$= r^2 - (\hat{e}\cdot\vec{r})^2 = a^2 \longleftarrow \text{see the drawing}$$

$\uparrow$ to define a particular cylinder

$$\nabla f = \underbrace{\nabla(r^2)}_{2r\hat{r} = 2\vec{r}} - 2(\hat{e}\cdot\vec{r})\underbrace{\nabla(\hat{e}\cdot\vec{r})}_{\hat{e}} = 2\left(\vec{r} - \hat{e}(\hat{e}\cdot\vec{r})\right)$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{\vec{r} - \hat{e}(\hat{e}\cdot\vec{r})}{a} \longleftarrow \text{obvious from the drawing}$$

The tangent plane at point $\vec{R}$ is defined by

$$\hat{n}\cdot\vec{r} = a, \quad \text{where } \hat{n} = \frac{\vec{R} - \hat{e}(\hat{e}\cdot\vec{R})}{a}.$$

This becomes

$$a^2 = \vec{r}\cdot\left[\vec{R} - \hat{e}(\hat{e}\cdot\vec{R})\right] = \vec{r}\cdot\vec{R} - (\hat{e}\cdot\vec{r})(\hat{e}\cdot\vec{R}).$$

The case of interest is $a = \sqrt{6}$ and $\vec{R} = 4\hat{x} + \hat{y} + \hat{z}$, so the equation for the tangent plane is

$$6 = 4x + y + z - \frac{1}{\sqrt{3}}(x+y+z)\frac{1}{\sqrt{3}}(4+1+1)$$

$$= 4x + y + z - 2(x+y+z)$$

$$= 2x - y - z,$$

as before.