5.3.

(a) Use Cartesian coördinates (x, y, z)

defn. of divergence

$$\nabla\cdot(\vec{F}\times\vec{G}) = \frac{\partial}{\partial x}(\vec{F}\times\vec{G})_x + \frac{\partial}{\partial y}(\vec{F}\times\vec{G})_y + \frac{\partial}{\partial z}(\vec{F}\times\vec{G})_z$$

def. of cross product

$$= \frac{\partial}{\partial x}\left(F_y G_z - F_z G_y\right) + \frac{\partial}{\partial y}\left(F_z G_x - F_x G_z\right) + \frac{\partial}{\partial z}\left(F_x G_y - F_y G_x\right)$$

product rule for derivatives

$$= \frac{\partial F_y}{\partial x} G_z - \frac{\partial F_z}{\partial x} G_y + \frac{\partial F_z}{\partial y} G_x - \frac{\partial F_x}{\partial y} G_z + \frac{\partial F_x}{\partial z} G_y - \frac{\partial F_y}{\partial z} G_x$$

$$+ F_y \frac{\partial G_z}{\partial x} - F_z \frac{\partial G_y}{\partial x} + F_z \frac{\partial G_x}{\partial y} - F_x \frac{\partial G_z}{\partial y} + F_x \frac{\partial G_y}{\partial z} - F_y \frac{\partial G_x}{\partial z}$$

gather terms

$$= G_x \underbrace{\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)}_{(\nabla\times\vec{F})_x} + G_y \underbrace{\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)}_{(\nabla\times\vec{F})_y} + G_z \underbrace{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)}_{(\nabla\times\vec{F})_z}$$

$$- F_x \underbrace{\left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right)}_{(\nabla\times\vec{G})_x} - F_y \underbrace{\left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right)}_{(\nabla\times\vec{G})_y} - F_z \underbrace{\left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)}_{(\nabla\times\vec{G})_z}$$

$$\boxed{\nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot\nabla\times\vec{F} - \vec{F}\cdot\nabla\times\vec{G}}$$

OR Use the antisymmetric symbol:

comma denotes partial derivative

$$\nabla\cdot(\vec{F}\times\vec{G}) = (\epsilon_{jkl} F_k G_l)_{,j}$$

$$= \epsilon_{jkl} F_{k,j} G_l + \epsilon_{jkl} F_k G_{l,j}$$

$$= G_l \underbrace{\epsilon_{ljk} F_{k,j}}_{(\nabla\times\vec{F})_l} - F_k \underbrace{\epsilon_{kjl} G_{l,j}}_{(\nabla\times\vec{G})_k}$$

$$= \vec{G}\cdot\nabla\times\vec{F} - \vec{F}\cdot\nabla\times\vec{G}$$

(b) $$\hat{e}\cdot\int_V \nabla\times\vec{A}\,d\tau = \int_V \hat{e}\cdot\nabla\times\vec{A}\,d\tau \quad (\hat{e} \text{ is constant})$$

$\uparrow$ arbitrary (constant) unit vector

$$= \int_V \nabla\cdot(\vec{A}\times\hat{e})\,d\tau \quad \left[\begin{array}{c}\text{part (a) and}\\ \nabla\times\hat{e}=0\end{array}\right]$$

$$= \oint_S \vec{A}\times\vec{a}\cdot\vec{da} \quad (\text{divergence theorem})$$

$$= \oint_S \vec{da}\times\vec{A}\cdot\hat{e} \quad \left(\begin{array}{c}\text{cyclic property of}\\ \text{vector triple product}\end{array}\right)$$

$$= \hat{e}\cdot\oint_S \vec{da}\times\vec{A} \quad (\hat{e} \text{ is constant})$$

$$\Longrightarrow \boxed{\int_V \nabla\times\vec{A}\,d\tau = \oint_S \vec{da}\times\vec{A}}$$

Because all components of the two vector integrals are equal — e.g., their components along $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$

(c) $$\oint_S \rho\hat{e}_\phi\times\vec{da} = -\oint_S \vec{da}\times\rho\hat{e}_\phi$$

$$= -\int_V \nabla\times(\rho\hat{e}_\phi)\,d\tau \quad \left(\begin{array}{c}\text{from part}\\ \text{(b)}\end{array}\right)$$

$$\nabla\times(\rho\hat{e}_\phi) = 2\hat{e}_z \longleftarrow$$ Calculate curl in <u>cylindrical</u> coördinates

$$\Longrightarrow \oint_S \rho\hat{e}_\phi\times\vec{da} = -2\hat{e}_z\int_V d\tau$$

$$= -2\hat{e}_z\,\frac{4}{3}\pi R^3$$

Volume of a sphere of radius R

$$= -\frac{8\pi}{3}R^3\hat{e}_z$$

(d) Direct evaluation of surface integral: Use spherical coördinates for everything at the start.

$$\oint_S \underset{\uparrow \atop R\sin\theta}{\rho}\, \hat{e}_\phi \times \underbrace{\overrightarrow{da}}_{R^2 \sin\theta\, d\theta\, d\phi\, \hat{e}_r} = \int R^3 \sin^2\theta\, d\theta\, d\phi\, \underbrace{\hat{e}_\phi \times \hat{e}_r}_{\hat{e}_\theta}$$

$$\therefore \oint_S \rho \hat{e}_\phi \times \overrightarrow{da} = R^3 \int \sin^2\theta\, d\theta\, d\phi\, \hat{e}_\theta$$

We have to worry about how $\hat{e}_\theta$ changes as we move from place to place on S in the integral. Write $\hat{e}_\theta$ in terms of the Cartesian basis vectors, which do not change from place to place:

$$\hat{e}_\theta = \cos\theta \cos\phi\, \hat{e}_x + \cos\theta \sin\phi\, \hat{e}_y - \sin\theta\, \hat{e}_z$$

The x and y components integrate to zero, because $\int_0^{2\pi} d\phi \cos\phi = \int_0^{2\pi} \sin\phi\, d\phi = 0$, so we have

$$\oint_S \rho \hat{e}_\phi \times \overrightarrow{da} = -R^3 \hat{e}_z \int_0^{\pi} d\theta \sin^3\theta \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\int_0^{\pi} d\theta \sin^3\theta = \int_0^{\pi} d\theta \sin\theta (1 - \cos^2\theta)$$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$

$$= \int_{-1}^{+1} du (1 - u^2) = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\therefore \boxed{\oint_S \rho\, \hat{e}_\phi \times \vec{da} = -\frac{8\pi}{3} R^3 \hat{e}_z}$$