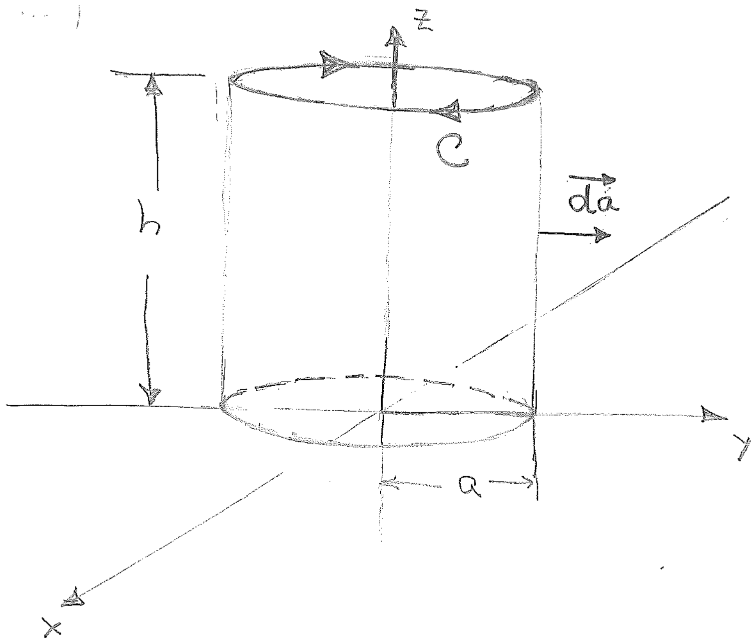

S - side and bottom of cylinder (like a trash can)

(b) $$\vec{A} = \overbrace{L\rho\,\hat{e}_\phi}^{-Ly\hat{e}_x + Lx\hat{e}_y} + M\overbrace{\rho\sin\phi}^{y}\,\hat{e}_z$$

Calculate $$\int_S \nabla\times\vec{A}\cdot\vec{da} \overset{\text{Stokes's theorem}}{=} \oint_C \vec{A}\cdot\underbrace{\vec{dl}}_{-a\,d\phi\,\vec{a}_\phi \text{ with } d\phi>0}$$

Note direction of $\vec{dl}$

$$= -a\int_0^{2\pi} A_\phi\, d\phi$$

$$= -a\int_0^{2\pi} La\, d\phi$$

$$\boxed{\int_S \nabla\times\vec{A}\cdot\vec{da} = -2\pi L a^2}$$

Surface integral calculations:

$$\nabla\times\vec{A} = \underbrace{\frac{\partial A_z}{\partial y}}_{M}\hat{e}_x + \left(\underbrace{\frac{\partial A_y}{\partial x}}_{+L} - \underbrace{\frac{\partial A_x}{\partial y}}_{-L}\right)\hat{e}_z = 2L\hat{e}_z + M\hat{e}_x$$
$$= 2L\hat{e}_z + M(\hat{e}_\rho\cos\phi - \hat{e}_\phi\sin\phi)$$

or $$\nabla\times\vec{A} = \underbrace{\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}}_{M\cos\phi}\hat{e}_\rho - \underbrace{\frac{\partial A_z}{\partial\rho}}_{M\sin\phi}\hat{e}_\phi + \underbrace{\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)}_{2L}\hat{e}_z = 2L\hat{e}_z + M\cos\phi\,\hat{e}_\rho - M\sin\phi\,\hat{e}_\phi$$

(a) $$\int_S \nabla\times\vec{A}\cdot\vec{da} = \int_{\text{side}} \nabla\times\vec{A}\cdot\underbrace{\vec{da}}_{a\,d\phi\,dz\,\hat{e}_\rho} + \int_{\text{bottom}} \nabla\times\vec{A}\cdot\underbrace{\vec{da}}_{-\rho\,d\rho\,d\phi\,\hat{e}_z}$$

$$= \underbrace{a\int_0^h dz \int_0^{2\pi} d\phi\, \underbrace{(\nabla\times\vec{A})_\rho}_{M\cos\phi}}_{0} - \underbrace{\int_0^a \rho\,d\rho \int_0^{2\pi} d\phi\, \underbrace{(\nabla\times\vec{A})_z}_{2L}}_{2L\pi a^2}$$

$$= -2\pi L a^2$$

(c) Stokes's theorem also says that the integral over the side and bottom is the same as the integral over the top with downward-pointing area element.

$$\int_S \nabla\times\vec{A}\cdot\vec{da} = \int_{\text{top}} \nabla\times\vec{A}\cdot\underbrace{\vec{da}}_{-\rho\,d\rho\,d\phi\,\hat{e}_z} = -2\pi L a^2$$

This is trivially the same as the integral over the bottom of the trash can and provides another reason why the integral over the sides has to vanish.