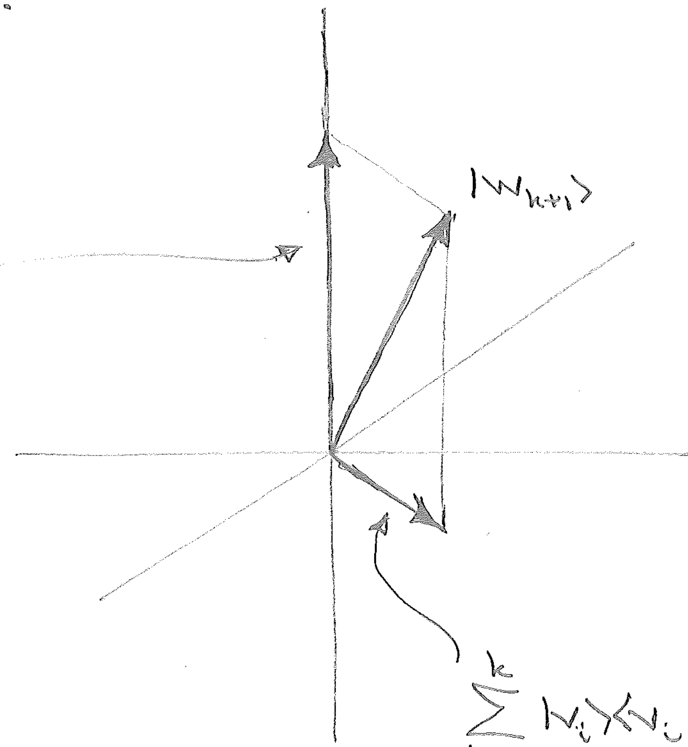
6.1 (a) Picture:



Horizontal plane is subspace spanned by $|w_1\rangle, \dots, |w_k\rangle$ and by $|v_1\rangle, \dots, |v_k\rangle$

$$\sum_{i=1}^{k} |v_i\rangle\langle v_i|w_{k+1}\rangle = \left(\begin{array}{l}\text{Component of} \\ |w_{k+1}\rangle \text{ in subspace} \\ \text{spanned by} \\ |v_1\rangle, \dots, |v_k\rangle\end{array}\right)$$

$$|w_{k+1}\rangle - \sum_{i=1}^{k} |v_i\rangle\langle v_i|w_{k+1}\rangle = \left(\begin{array}{l}\text{component of } |w_{k+1}\rangle \\ \text{orthogonal to subspace} \\ \text{spanned by } |v_1\rangle, \dots, |v_k\rangle\end{array}\right)$$

Normalize this to get $|v_{k+1}\rangle$

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} |v_i\rangle\langle v_i|w_{k+1}\rangle}{\left\| |w_{k+1}\rangle - \sum_{i=1}^{k} |v_i\rangle\langle v_i|w_{k+1}\rangle \right\|}$$

$$= \Big[\langle w_{k+1}|w_{k+1}\rangle - \sum_{i=1}^{k} \langle w_{k+1}|v_i\rangle\langle v_i|w_{k+1}\rangle - \sum_{i=1}^{k} \langle w_{k+1}|v_i\rangle\langle v_i|w_{k+1}\rangle + \sum_{i,j=1}^{k} \langle w_{k+1}|v_i\rangle\underbrace{\langle v_i|v_j\rangle}^{\delta_{ij}}\langle v_j|w_{k+1}\rangle \Big]$$

$$= \left[\langle w_{k+1} | w_{k+1} \rangle - \sum_{i=1}^{k} |\langle w_{k+1} | v_i \rangle|^2 \right]^{1/2}$$

$$= \left[\langle w_{k+1} | \left(1 - \sum_{i=1}^{k} |v_i\rangle\langle v_i| \right) | w_{k+1} \rangle \right]^{1/2}$$

Algebra: It is clear that the states $|v_i\rangle$ are normalized, so all we need to show is that each state is $\perp$ to the previous states in the list. Inductively, we assume $\langle v_i | v_j \rangle = \delta_{ij}$ $i, j \leq k$, and we <u>show</u> $\langle v_i | v_{k+1} \rangle = 0$ for $i \leq k$.

$$\langle v_i | v_{k+1} \rangle = \frac{\langle v_i | w_{k+1} \rangle - \sum_{j=1}^{k} \overbrace{\langle v_i | v_j \rangle}^{\delta_{ij}} \langle v_j | w_{k+1} \rangle}{\| \qquad \|}$$

$$= \frac{\langle v_i | w_{k+1} \rangle - \langle v_i | w_{k+1} \rangle}{\| \qquad \|}$$

$$= 0.$$

(b) Vectors $|e_j\rangle$, $j=1,\dots,d$, in a d-dimensional vector space, satisfying the completeness relation

$$I = \sum_{j=1}^{d} |e_j\rangle\langle e_j|$$

① Any vector $|\psi\rangle$ can be expanded in terms of the vectors $|e_j\rangle$, since

$$|\psi\rangle = I|\psi\rangle = \sum_{j=1}^{d} |e_j\rangle\langle e_j|\psi\rangle.$$

This means that the vectors $|e_j\rangle$ span the vector space.

② For d vectors to span, they must be linearly independent, so the above expansion is unique.

③ Expand $|e_j\rangle$:

$$|e_j\rangle = \sum_{k=1}^{d} |e_k\rangle\langle e_k|e_j\rangle$$

Uniqueness requires that $\langle e_k|e_j\rangle$; i.e, the vectors $|e_j\rangle$ are an orthonormal basis.