6.2. $|+,z\rangle \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$, $|-,z\rangle \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$

$$\sigma_z = |+,z\rangle\langle +,z| - |-,z\rangle\langle -,z| \longleftrightarrow \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

(a)

Kets:

$$|+,y\rangle = \frac{1}{\sqrt{2}}\big(|+,z\rangle + i|-,z\rangle\big) \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

$$|-,y\rangle = \frac{1}{\sqrt{2}}\big(|+,z\rangle - i|-,z\rangle\big) \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

These are z-representations

$$\langle +,y| = \frac{1}{\sqrt{2}}\big(\langle +,z| - i\langle -,z|\big) \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}$$

$$\langle -,y| = \frac{1}{\sqrt{2}}\big(\langle +,z| + i\langle -,z|\big) \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}$$

Orthonormality:

$$\langle +,y|+,y\rangle = \frac{1}{2}(1+1) = 1$$

$$\langle -,y|-,y\rangle = \frac{1}{2}(1+1) = 1$$

$$\langle -,y|+,y\rangle = \frac{1}{2}(1-1) = 0$$

Inversion:

Kets

$$|+,z\rangle = \frac{1}{\sqrt{2}}\big(|+,y\rangle + |-,y\rangle\big)$$

$$|-,z\rangle = -\frac{i}{\sqrt{2}}\big(|+,y\rangle - |-,y\rangle\big)$$

Bras

$$\langle +,z| = \frac{1}{\sqrt{2}}\big(\langle +,y| + \langle -,y|\big)$$

$$\langle -,z| = \frac{i}{\sqrt{2}}\big(\langle +,y| - \langle -,y|\big)$$

(b) $\sigma_y = -i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z| \longleftrightarrow \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

$$\sigma_y|+,y\rangle = \big(-i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|\big)\frac{1}{\sqrt{2}}\big(|+,z\rangle + i|-,z\rangle\big)$$

$$= \frac{1}{\sqrt{2}}\big(|+,z\rangle + i|-,z\rangle\big)$$

$$= |+,y\rangle$$ Eigenvector with eigenvalue $+1$

$$\sigma_y|-,y\rangle = \big(-i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|\big)\frac{1}{\sqrt{2}}\big(|+,z\rangle - i|-,z\rangle\big)$$

$$= \frac{1}{\sqrt{2}}\big(-|+,z\rangle + i|-,z\rangle\big)$$

$$= -|-,y\rangle$$ Eigenvector with eigenvalue -1

<u>OR</u> use the matrix representation

$$\sigma_y|\pm,y\rangle \longleftrightarrow \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\pm 1\\ i\end{pmatrix} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix} \longleftrightarrow \pm|\pm,y\rangle$$

(c)

$$\sigma_z = |+,z\rangle\langle +,z| - |-,z\rangle\langle -,z|$$
$$= \tfrac{1}{2}\big(|+,y\rangle + |-,y\rangle\big)\big(\langle +,y| + \langle -,y|\big) - \tfrac{1}{2}\big(|+,y\rangle - |-,y\rangle\big)\big(\langle +,y| - \langle -,y|\big)$$

$$\sigma_z = |+,y\rangle\langle -,y| + |-,y\rangle\langle +,y| \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ This is a y representation

$$\sigma_y = -i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|$$
$$= \tfrac{1}{2}\big(|+,y\rangle + |-,y\rangle\big)\big(\langle +,y| - \langle -,y|\big) + \tfrac{1}{2}\big(|+,y\rangle - |-,y\rangle\big)\big(\langle +,y| + \langle -,y|\big)$$

$$\sigma_y = |+,y\rangle\langle +,y| - |-,y\rangle\langle -,y| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ This is a y representation

This result is obvious from our calculation that $|\pm,y\rangle$ is a eigenvector of σ_y with eigenvalue ± 1.

(d) $c_\pm = \langle \pm,z|\psi\rangle$, $\quad d_\pm = \langle \pm,y|\psi\rangle$

There are several ways to get the matrix V. One is to write

$$d_\xi = \langle \xi,y|\psi\rangle = \sum_\eta \underbrace{\langle \xi,y|\eta,z\rangle}_{V_{\xi\eta}} \underbrace{\langle \eta,z|\psi\rangle}_{c_\eta} \leftrightarrow \begin{pmatrix} d_+ \\ d_- \end{pmatrix} = V \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

and thus to identify

$$V_{\xi\eta} = \langle \xi,y|\eta,z\rangle,$$

so

$$\|V_{\xi\eta}\| = \begin{pmatrix} \langle +,y|+,z\rangle & \langle +,y|-,z\rangle \\ \langle -,y|+,z\rangle & \langle -,y|-,z\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

We could also say

$$|\eta,z\rangle = \sum_\xi |\xi,y\rangle \underbrace{\langle \xi,y|\eta,z\rangle}_{V_{\xi\eta}}$$

and read the matrix elements off our expressions in (a),

$$V_{++} = \langle +,y|+,z\rangle = \frac{1}{\sqrt{2}} \qquad V_{+-} = \langle +,y|-,z\rangle = -\frac{i}{\sqrt{2}}$$
$$V_{-+} = \langle -,y|+,z\rangle = \frac{1}{\sqrt{2}} \qquad V_{--} = \langle -,y|-,z\rangle = \frac{i}{\sqrt{2}}$$

giving the same V as above. So

$$d_\xi = \sum_\eta V_{\xi\eta} c_\eta \qquad \leftrightarrow \begin{pmatrix} d_+ \\ d_- \end{pmatrix} = V \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

Complex conjugate $$d_\xi^* = \sum_\eta V_{\xi\eta}^* c_\eta^* = \sum_\eta c_\eta^* (V^\dagger)_{\eta\xi} \leftrightarrow (d_+^* \;\; d_-^*) = (c_+^* \;\; c_-^*) V^\dagger$$

Inverse $$c_\eta = \langle \eta,z|\psi\rangle = \sum_\xi \underbrace{\langle \eta,z|\xi,y\rangle}_{V_{\xi\eta}^* = (V^\dagger)_{\eta\xi}} \underbrace{\langle \xi,y|\psi\rangle}_{d_\xi} = \sum_\xi (V^\dagger)_{\eta\xi} d_\xi$$

$$\longleftrightarrow \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = V^\dagger \begin{pmatrix} d_+ \\ d_- \end{pmatrix}$$

Complex conjugate of inverse $\quad c_\eta^* = \sum_\xi d_\xi^* V_{\xi\eta} \quad \longleftrightarrow \quad (c_+^* \; c_-^*) = (d_+^* \; d_-^*) V$

(e) $|\xi, y\rangle = U|\xi, z\rangle \Rightarrow \underbrace{\langle \eta, z|\xi, y\rangle}_{V_{\xi\eta}^* = (V^\dagger)_{\eta\xi}} = \underbrace{\langle \eta, z|U|\xi, z\rangle}_{U_{\eta\xi}}$ z matrix elements of U

$$U_{\eta\xi} = (V^\dagger)_{\eta\xi} \longleftrightarrow \| U_{\eta\xi} \| = V^\dagger$$

The matrix representation of U in the z basis is the inverse of V, because an active transformation is the inverse of the corresponding passive transformation.