7.1 $\sigma_z = |+,z\rangle\langle +,z| - |-,z\rangle\langle -,z| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Eigenvectors $|\pm,z\rangle$ with eigenvalues ± 1: $\sigma_z|\pm,z\rangle = \pm|\pm,z\rangle$

$\sigma_x = |+,z\rangle\langle -,z| + |-,z\rangle\langle +,z| \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_y = -i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z| \longleftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

(a) $\sigma_x^2 = \left(|+,z\rangle\langle -,z| + |-,z\rangle\langle +,z|\right)\left(|+,z\rangle\langle -,z| + |-,z\rangle\langle +,z|\right)$

$= |+,z\rangle\langle +,z| + |-,z\rangle\langle -,z|$

$$\boxed{\sigma_x^2 = I}$$

$\sigma_x\sigma_y = \left(|+,z\rangle\langle -,z| + |-,z\rangle\langle +,z|\right)\left(-i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|\right)$

$= i\underbrace{\left(|+,z\rangle\langle +,z| - |-,z\rangle\langle -,z|\right)}_{\sigma_z}$

$$\boxed{\sigma_x\sigma_y = i\sigma_z}$$

$\sigma_x\sigma_z = \left(|+,z\rangle\langle -,z| + |-,z\rangle\langle +,z|\right)\left(|+,z\rangle\langle +,z| - |-,z\rangle\langle -,z|\right)$

$= \underbrace{-|+,z\rangle\langle -,z| + |-,z\rangle\langle +,z|}_{-i\sigma_y}$

$$\boxed{\sigma_x\sigma_z = -i\sigma_y}$$

$\sigma_y\sigma_x = \left(-i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|\right)\left(|+,z\rangle\langle -,z| + |-,z\rangle\langle +,z|\right)$

$= \underbrace{-i|+,z\rangle\langle +,z| + i|-,z\rangle\langle -,z|}_{-i\sigma_z}$

$$\boxed{\sigma_y\sigma_x = -i\sigma_z}$$

②

$$\sigma_y^2 = \left(-i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|\right)\left(-i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|\right)$$

$$= \underbrace{|+,z\rangle\langle +,z| + |-,z\rangle\langle -,z|}_{I}$$

$$\boxed{\sigma_y^2 = I}$$

$$\sigma_y\sigma_z = \left(-i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|\right)\left(|+,z\rangle\langle +,z| - |-,z\rangle\langle -,z|\right)$$

$$= \underbrace{i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|}_{i\sigma_x}$$

$$\boxed{\sigma_y\sigma_z = i\sigma_x}$$

$$\sigma_z\sigma_x = \left(|+,z\rangle\langle +,z| - |-,z\rangle\langle -,z|\right)\left(|+,z\rangle\langle -,z| + |-,z\rangle\langle +,z|\right)$$

$$= \underbrace{|+,z\rangle\langle -,z| - |-,z\rangle\langle +,z|}_{i\sigma_y}$$

$$\boxed{\sigma_z\sigma_x = i\sigma_y}$$

$$\sigma_z\sigma_y = \left(|+,z\rangle\langle +,z| - |-,z\rangle\langle -,z|\right)\left(-i|+,z\rangle\langle -,z| + i|-,z\rangle\langle +,z|\right)$$

$$= \underbrace{-i|+,z\rangle\langle -,z| - i|-,z\rangle\langle +,z|}_{-i\sigma_x}$$

$$\boxed{\sigma_z\sigma_y = -i\sigma_x}$$

$$\sigma_z^2 = \left(|+,z\rangle\langle +,z| - |-,z\rangle\langle -,z|\right)\left(|+,z\rangle\langle +,z| - |-,z\rangle\langle -,z|\right)$$

$$= \underbrace{|+,z\rangle\langle +,z| + |-,z\rangle\langle -,z|}_{I}$$

$$\boxed{\sigma_z^2 = I}$$

All this can be summarized as

$$\sigma_j \sigma_k = \delta_{jk} I + i \epsilon_{jkl} \sigma_l \iff \begin{array}{l} \sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} I \\ [\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l \end{array}$$

(b) Eigenvectors and eigenvalues of σ_x and σ_y:

① $\sigma_x |\lambda\rangle = \lambda |\lambda\rangle$

Characteristic equation: $0 = \det(\sigma_x - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1$

$$\Rightarrow \lambda = \pm 1$$

$\lambda = +1$: $\begin{pmatrix} a \\ b \end{pmatrix} = \sigma_x \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \Rightarrow a = b$

Normalization $\Rightarrow 1 = |a|^2 + |b|^2 = 2|a|^2 \Rightarrow |a| = \frac{1}{\sqrt{2}}$

Choose a to be real and positive, so $a = \frac{1}{\sqrt{2}} = b$

The eigenvector is

$\|+,x\rangle = \frac{1}{\sqrt{2}}(\|+,z\rangle + \|-,z\rangle)$	eigenvalue $+1$

$\lambda = -1$: $-\begin{pmatrix} a \\ b \end{pmatrix} = \sigma_x \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \Rightarrow b = -a$

Normalization $\Rightarrow 1 = |a|^2 + |b|^2 = 2|a|^2 \Rightarrow |a| = \frac{1}{\sqrt{2}}$

Choose a to be real and positive, so $a = \frac{1}{\sqrt{2}} = -b$

The eigenvector is

$\|-,x\rangle = \frac{1}{\sqrt{2}}(\|+,z\rangle - \|-,z\rangle)$	eigenvalue -1

Similarly, for σ_y, the eigenvectors are

$\|+,y\rangle = \frac{1}{\sqrt{2}}(\|+,z\rangle + i\|-,z\rangle)$	eigenvalue $+1$
$\|-,y\rangle = \frac{1}{\sqrt{2}}(\|+,z\rangle - i\|-,z\rangle)$	eigenvalue -1

$$\boxed{\begin{aligned} \sigma_x &= |+,x\rangle\langle +,x| - |-,x\rangle\langle -,x| \\ \sigma_y &= |+,y\rangle\langle +,y| - |-,y\rangle\langle -,y| \\ \sigma_z &= |+,z\rangle\langle +,z| - |-,z\rangle\langle -,z| \end{aligned}}$$

σ_x, σ_y, and σ_z look the same in their own eigenbasis, because they have the same eigenvalues $+1$ and -1.

(c) State $|\psi\rangle = |+,z\rangle$

Measure σ_z : $p_+ = |\langle +,z|+,z\rangle|^2 = 1$, $p_- = |\langle -,z|+,z\rangle|^2 = 0$

Measure σ_x : $p_+ = |\langle +,x|+,z\rangle|^2 = \frac{1}{2}$, $p_- = |\langle -,x|+,z\rangle|^2 = \frac{1}{2}$

Measure σ_y : $p_+ = |\langle +,y|+,z\rangle|^2 = \frac{1}{2}$, $p_- = |\langle -,y|+,z\rangle|^2 = \frac{1}{2}$

(d) State $|\psi\rangle = |+,x\rangle$

Measure σ_z : $p_+ = |\langle +,z|+,x\rangle|^2 = \frac{1}{2}$, $p_- = |\langle -,z|+,x\rangle|^2 = \frac{1}{2}$

Measure σ_x : $p_+ = |\langle +,x|+,x\rangle|^2 = 1$, $p_- = |\langle -,x|+,x\rangle|^2 = 0$

Measure σ_y : $p_+ = |\underbrace{\langle +,y|+,x\rangle}|^2 = \frac{1}{2}$, $p_- = |\underbrace{\langle -,y|+,x\rangle}|^2 = \frac{1}{2}$

$$= \frac{1}{\sqrt{2}}\left(\langle +,z| - i\langle -,z|\right)\frac{1}{\sqrt{2}}\left(|+,z\rangle + |-,z\rangle\right)$$
$$= \frac{1}{2}(1-i)$$
$$= \frac{1}{\sqrt{2}}e^{-i\pi/4}$$

$$= \frac{1}{\sqrt{2}}\left(\langle +,z| + i\langle -,z|\right)\frac{1}{\sqrt{2}}\left(|+,z\rangle + |-,z\rangle\right)$$
$$= \frac{1}{2}(1+i)$$
$$= \frac{1}{\sqrt{2}}e^{+i\pi/4}$$

(e) State $|\psi\rangle = |+,y\rangle$

Measure σ_z : $p_+ = |\langle +,z|+,y\rangle|^2 = \frac{1}{2}$, $p_- = |\langle -,z|+,y\rangle|^2 = \frac{1}{2}$

Measure σ_x : $p_+ = |\langle +,x|+,y\rangle|^2 = \frac{1}{2}$, $p_- = |\langle -,x|+,y\rangle|^2 = \frac{1}{2}$

Measure σ_y : $p_+ = |\langle +,y|+,y\rangle|^2 = 1$, $p_- = |\langle -,y|+,y\rangle|^2 = 0$

(f) No matter what you measure, the probability to find the system in the state $|r\rangle$ comes from the following:

$$p_r = \frac{1}{2}\underbrace{\begin{pmatrix}\text{probability for } |r\rangle \\ \text{given system in } |+,z\rangle\end{pmatrix}}_{|\langle r|+,z\rangle|^2} + \frac{1}{2}\underbrace{\begin{pmatrix}\text{probability for } |r\rangle \\ \text{given system in } |-,x\rangle\end{pmatrix}}_{|\langle r|-,x\rangle|^2}$$

Measure σ_z: $P_+ = \frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{2} = \frac{3}{4}$

$P_- = \frac{1}{2}\times 0 + \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

Measure σ_x: $P_+ = \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times 0 = \frac{1}{4}$

$P_- = \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times 1 = \frac{3}{4}$

Measure σ_y: $P_+ = \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2} = \frac{1}{2}$

$P_- = \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2} = \frac{1}{2}$

(g) According to (f),

$$P_r = \frac{1}{2}\left(|\langle r|+,z\rangle|^2 + |\langle r|-,x\rangle|^2\right)$$

$$= \frac{1}{2}\left(\langle r|+,z\rangle\langle +,z|r\rangle + \langle r|-,x\rangle\langle -,x|r\rangle\right)$$

$$= \langle r|\underbrace{\frac{1}{2}\left(|+,z\rangle\langle +,z| + |-,x\rangle\langle -,x|\right)}_{\rho}|r\rangle$$

All probabilities are matrix elements of the density operator $\rho = \frac{1}{2}\left(|+,z\rangle\langle +,z| + |-,x\rangle\langle -,x|\right)$. To diagonalize ρ, we need to write in a single orthonormal basis, so we put in

$$|-,x\rangle = \frac{1}{\sqrt{2}}\left(|+,z\rangle - |-,z\rangle\right), \quad \langle -,x| = \frac{1}{\sqrt{2}}\left(\langle +,z| - \langle -,z|\right)$$

$$|-,x\rangle\langle -,x| = \frac{1}{2}\Big(|+,z\rangle\langle +,z| + |-,z\rangle\langle -,z| - |+,z\rangle\langle -,z| - |-,z\rangle\langle +,z|\Big)$$

So

$$\rho = \frac{1}{2}|+,z\rangle\langle +,z| + \frac{1}{4}\Big(|+,z\rangle\langle +,z| + |-,z\rangle\langle -,z| - |+,z\rangle\langle -,z| - |-,z\rangle\langle +,z|\Big)$$

$$\rho = \frac{3}{4}|+,z\rangle\langle +,z| + \frac{1}{4}|-,z\rangle\langle -,z| - \frac{1}{4}|+,z\rangle\langle -,z| - \frac{1}{4}|-,z\rangle\langle +,z| \longleftrightarrow \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 1/4 \end{pmatrix}$$

Diagonalizing ρ: $\rho|\lambda\rangle = \lambda|\lambda\rangle$

Characteristic equation: $0 = \det(\rho - \lambda I)$

$$= \det\begin{pmatrix} 3/4-\lambda & -1/4 \\ -1/4 & 1/4-\lambda \end{pmatrix}$$

$$= \left(\lambda - \frac{3}{4}\right)\left(\lambda - \frac{1}{4}\right) - \frac{1}{16}$$

$$= \lambda^2 - \lambda + \frac{1}{8}$$

Solving, $\lambda = \dfrac{1 \pm \sqrt{1 - 1/2}}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{2}}{4} = \dfrac{1}{4}(2 \pm \sqrt{2})$

$\lambda = \frac{1}{4}(2+\sqrt{2})$: $\begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 1/4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{4}(2+\sqrt{2})\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$ $b = -(\sqrt{2}-1)a$, $a = -(\sqrt{2}+1)b$

$$|\lambda\rangle = a\begin{pmatrix} 1 \\ -(\sqrt{2}-1) \end{pmatrix} = b\begin{pmatrix} -(\sqrt{2}+1) \\ 1 \end{pmatrix}$$

It is useful to note that

$$\cos\frac{\pi}{8} = \frac{1}{\sqrt{2}}\left(1 + \cos\frac{\pi}{4}\right)^{1/2} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{\sqrt{2}}\right)^{1/2} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$\sin\frac{\pi}{8} = \frac{1}{\sqrt{2}}\left(1 - \cos\frac{\pi}{4}\right)^{1/2} = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{2}}\right)^{1/2} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

$$\tan\frac{\pi}{8} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{2}-1$$

This allows us to write $|\lambda\rangle$ as the normalized state

$$\boxed{|\lambda\rangle = \cos\frac{\pi}{8}|+,z\rangle - \sin\frac{\pi}{8}|-,z\rangle \quad \begin{matrix}\text{eigenvalue} \\ \lambda = \frac{1}{4}(2+\sqrt{2})\end{matrix}}$$

The eigenstate with eigenvalue $\frac{1}{4}(2-\sqrt{2})$ is orthogonal to $|\lambda\rangle$, so it is

$$\boxed{\sin\frac{\pi}{8}|+,z\rangle + \cos\frac{\pi}{8}|-,z\rangle \quad \begin{matrix}\text{eigenvalue} \\ \lambda = \frac{1}{4}(2-\sqrt{2})\end{matrix}}$$