7.2.

$$J_z = \sum_m m\hbar\, |m,z\rangle\langle m,z| = \hbar\left(|1,z\rangle\langle 1,z| - |-1,z\rangle\langle -1,z|\right)$$

$$\longleftrightarrow \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

J_z is diagonal in this basis so the states $|m,z\rangle$ are the eigenvectors of J_z.

$$J_x = \frac{\hbar}{\sqrt{2}}\left(|1,z\rangle\langle 0,z| + |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| + |-1,z\rangle\langle 0,z|\right)$$

$$\longleftrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_y = -\frac{i\hbar}{\sqrt{2}}\left(|1,z\rangle\langle 0,z| - |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| - |-1,z\rangle\langle 0,z|\right)$$

$$\longleftrightarrow -\frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Just use the orthonormality of the basis $|m,z\rangle$; this is easier than multiplying matrices.

(a) ✓ $$J_z^2 = \hbar^2\left(|1,z\rangle\langle 1,z| + |-1,z\rangle\langle -1,z|\right) \longleftrightarrow \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J_x^2 = \frac{\hbar^2}{2}\left(|1,z\rangle\langle 0,z| + |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| + |-1,z\rangle\langle 0,z|\right)$$

$$\times\left(|1,z\rangle\langle 0,z| + |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| + |-1,z\rangle\langle 0,z|\right)$$

✓ $$J_x^2 = \frac{\hbar^2}{2}\left(|1,z\rangle\langle 1,z| + |1,z\rangle\langle -1,z| + 2|0,z\rangle\langle 0,z| + |-1,z\rangle\langle 1,z| + |-1,z\rangle\langle -1,z|\right)$$

$$\longleftrightarrow \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

✓ $$J_y^2 = \frac{\hbar^2}{2}\left(|1,z\rangle\langle 1,z| - |1,z\rangle\langle -1,z| + 2|0,z\rangle\langle 0,z| - |-1,z\rangle\langle 1,z| + |-1,z\rangle\langle -1,z|\right)$$

$$\longleftrightarrow \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$J_x^2 + J_y^2 + J_z^2 = 2\hbar^2 \underbrace{\left(|1,z\rangle\langle 1,z| + |0,z\rangle\langle 0,z| + |-1,z\rangle\langle -1,z| \right)}_{I}$$

$$\longleftrightarrow 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J_x J_y = -\frac{i\hbar^2}{2} \left(|1,z\rangle\langle 0,z| + |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| + |-1,z\rangle\langle 0,z| \right)$$
$$\times \left(|1,z\rangle\langle 0,z| - |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| - |-1,z\rangle\langle 0,z| \right)$$

$$J_x J_y = \frac{i\hbar^2}{2} \left(|1,z\rangle\langle 1,z| - |1,z\rangle\langle -1,z| + |-1,z\rangle\langle 1,z| - |-1,z\rangle\langle -1,z| \right)$$

$$\longleftrightarrow \frac{i\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$J_y J_x = \frac{i\hbar^2}{2} \left(-|1,z\rangle\langle 1,z| - |1,z\rangle\langle -1,z| + |-1,z\rangle\langle 1,z| + |-1,z\rangle\langle -1,z| \right)$$

$$\longleftrightarrow \frac{i\hbar^2}{2} \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

This also follows from $J_y J_x = (J_x J_y)^\dagger$

$$[J_x, J_y] = J_x J_y - J_y J_x = i\hbar^2 \left(|1,z\rangle\langle 1,z| - |-1,z\rangle\langle -1,z| \right) = i\hbar J_z$$

$$J_z J_x = \frac{\hbar^2}{\sqrt{2}} \left(|1,z\rangle\langle 1,z| - |-1,z\rangle\langle -1,z| \right)$$
$$\times \left(|1,z\rangle\langle 0,z| + |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| + |-1,z\rangle\langle 0,z| \right)$$

$$J_z J_x = \frac{\hbar^2}{\sqrt{2}} \left(|1,z\rangle\langle 0,z| - |-1,z\rangle\langle 0,z| \right) \longleftrightarrow \frac{\hbar^2}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$J_x J_z = \frac{\hbar^2}{\sqrt{2}} \left(|0,z\rangle\langle 1,z| - |0,z\rangle\langle -1,z| \right) \longleftrightarrow \frac{\hbar^2}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This also follows from $J_x J_z = (J_z J_x)^\dagger$

$$[J_z, J_x] = J_z J_x - J_x J_z \longleftrightarrow \frac{\hbar^2}{\sqrt{2}} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{i\frac{\sqrt{2}}{\hbar} J_y} \longleftrightarrow i\hbar J_y$$

$$[J_z, J_x] = i\hbar J_y$$

$$J_y J_z = -\frac{i\hbar^2}{\sqrt{2}}\Big(|1,z\rangle\langle 0,z| - |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| - |-1,z\rangle\langle 0,z|\Big)$$
$$\times\Big(|1,z\rangle\langle 1,z| - |-1,z\rangle\langle -1,z|\Big)$$

✓ $$J_y J_z = \frac{i\hbar^2}{\sqrt{2}}\Big(|0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z|\Big) \longleftrightarrow \frac{i\hbar^2}{\sqrt{2}}\begin{pmatrix}0&0&0\\1&0&1\\0&0&0\end{pmatrix}$$

✓ $$J_z J_y = \frac{i\hbar^2}{\sqrt{2}}\Big(-|1,z\rangle\langle 0,z| - |-1,z\rangle\langle 0,z|\Big) \longleftrightarrow \frac{i\hbar^2}{\sqrt{2}}\begin{pmatrix}0&-1&0\\0&0&0\\0&-1&0\end{pmatrix}$$

This also follows from $J_z J_y = (J_y J_z)^\dagger$

$$[J_y, J_z] = J_y J_z - J_z J_y \longleftrightarrow \frac{i\hbar^2}{\sqrt{2}}\underbrace{\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}}_{\frac{\sqrt{2}}{\hbar}J_x} \longleftrightarrow i\hbar J_x$$

✓ $[J_y, J_z] = i\hbar J_x$

The three commutators, plus their negatives, are equivalent to $[J_j, J_k] = i\hbar\,\epsilon_{jkl} J_l$.

(b) Let's diagonalize J_x in the official way: $J_x|\lambda\rangle = \lambda|\lambda\rangle$

$$0 = \det(J_x - \lambda I) \Longrightarrow 0 = \det\begin{pmatrix}-\lambda & \hbar/\sqrt{2} & 0\\ \hbar/\sqrt{2} & -\lambda & \hbar/\sqrt{2}\\ 0 & \hbar/\sqrt{2} & -\lambda\end{pmatrix}$$

$$= -\lambda(\lambda^2 - \hbar^2/2) + \lambda\hbar^2/2$$

$$= \lambda(\hbar^2 - \lambda^2)$$

$$\Rightarrow \lambda = 0, \pm\hbar$$

① $\lambda = 0$: $$0 = J_x|\lambda\rangle = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} \Rightarrow b=0,\ a+c=0$$

$$|\lambda\rangle = a\big(|1,z\rangle - |-1,z\rangle\big)$$

Normalization $\Rightarrow |a| = \frac{1}{\sqrt{2}}$. Choose phase so that a is real and positive, i.e., $a = 1/\sqrt{2}$

This is the eigenstate $|0,x\rangle = \frac{1}{\sqrt{2}}\left(|1,z\rangle - |-1,z\rangle\right)$

② $\lambda = +\hbar$: $\hbar|\lambda\rangle = J_x|\lambda\rangle \iff \sqrt{2}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\iff \sqrt{2}\,a = b = \sqrt{2}\,c,\quad \sqrt{2}\,b = a + c$$

$$\iff a = c = \frac{1}{\sqrt{2}}\,b$$

$|\lambda\rangle = a\left(|1,z\rangle + \sqrt{2}|0,z\rangle + |-1,z\rangle\right)$

Normalization $\Rightarrow 1 = 4a^2 \Rightarrow |a| = \frac{1}{2}$

Choose phase so a is real and positive, i.e, $a = \frac{1}{2}$

This is the eigenstate $|1,x\rangle = \frac{1}{2}\left(|1,z\rangle + \sqrt{2}|0,z\rangle + |-1,z\rangle\right)$

③ $\lambda = -\hbar$: $-\hbar|\lambda\rangle = J_x|\lambda\rangle \iff -\sqrt{2}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\iff \sqrt{2}\,a = -b = \sqrt{2}\,c,\quad \sqrt{2}\,b = -a - c$$

$$\iff -a = -c = \frac{1}{\sqrt{2}}\,b$$

$|\lambda\rangle = a\left(|1,z\rangle - \sqrt{2}|0,z\rangle + |-1,z\rangle\right)$

Normalization $\Rightarrow 1 = 4a^2 \Rightarrow |a| = \frac{1}{2}$

Choose phase so a is real and positive, i.e, $a = \frac{1}{2}$

This is the eigenstate $|-1,x\rangle = \frac{1}{2}\left(|1,z\rangle - \sqrt{2}|0,z\rangle + |-1,z\rangle\right)$

There is a cleverer way to do this. We'll do J_x this cleverer way and J_y at the same time.

$$J_x = \frac{\hbar}{\sqrt{2}}\left(|1,z\rangle\langle 0,z| + |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| + |-1,z\rangle\langle 0,z|\right)$$

$$= \frac{\hbar}{\sqrt{2}}\Big[\underbrace{\left(|1,z\rangle + |-1,z\rangle\right)}_{\equiv\sqrt{2}|+\rangle}\langle 0,z| + |0,z\rangle\underbrace{\left(\langle 1,z| + \langle -1,z|\right)}_{\sqrt{2}\langle +|}\Big]$$

$$= \hbar\left(|+\rangle\langle 0,z| + |0,z\rangle\langle +|\right)$$

$$J_y = -\frac{i\hbar}{\sqrt{2}}\Big(|1,z\rangle\langle 0,z| - |0,z\rangle\langle 1,z| + |0,z\rangle\langle -1,z| - |-1,z\rangle\langle 0,z|\Big)$$

$$= -\frac{i\hbar}{\sqrt{2}}\Big[\underbrace{\big(|1,z\rangle - |-1,z\rangle\big)}_{\equiv\sqrt{2}\,|-\rangle}\langle 0,z| - |0,z\rangle\underbrace{\big(\langle 1,z| - \langle -1,z|\big)}_{\sqrt{2}\,\langle -|}\Big]$$

$$= \hbar\Big(-i|-\rangle\langle 0,z| + i|0,z\rangle\langle -|\Big)$$

The states $|+\rangle$, $|0,z\rangle$, $|-\rangle$ are an orthonormal basis, and we are now just a step from diagonalizing.

$$\begin{cases} J_x|+\rangle = \hbar|0,z\rangle \\ J_x|0,z\rangle = \hbar|+\rangle \end{cases}$$

$$J_x|-\rangle = 0$$

this is the eigenstate $|0,x\rangle$

Whenever you have an exchange like this, the eigenstates are

$\frac{1}{\sqrt{2}}\big(|+\rangle + |0,z\rangle\big)$ eigenvalue $+\hbar$

$\frac{1}{\sqrt{2}}\big(|+\rangle - |0,z\rangle\big)$ eigenvalue $-\hbar$

$$\begin{cases} J_y|-\rangle = i\hbar|0,z\rangle \\ J_y|0,z\rangle = -i\hbar|-\rangle \end{cases}$$

$$J_y|+\rangle = 0$$

this is the eigenstate $|0,y\rangle$

$\frac{1}{\sqrt{2}}\big(|-\rangle + i|0,z\rangle\big)$ eigenvalue $+\hbar$

$\frac{1}{\sqrt{2}}\big(|-\rangle - i|0,z\rangle\big)$ eigenvalue $-\hbar$

J_x:

$|1,x\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle + |0,z\rangle\big) = \frac{1}{2}\big(|1,z\rangle + \sqrt{2}|0,z\rangle + |-1,z\rangle\big)$ eigenvalue $\hbar$

$|0,x\rangle = |-\rangle = \frac{1}{\sqrt{2}}\big(|1,z\rangle - |-1,z\rangle\big)$ eigenvalue 0

$|-1,x\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle - |0,z\rangle\big) = \frac{1}{2}\big(|1,z\rangle - \sqrt{2}|0,z\rangle + |-1,z\rangle\big)$ eigenvalue $-\hbar$

J_y:

$|1,y\rangle = \frac{1}{\sqrt{2}}\big(|-\rangle + i|0,z\rangle\big) = \frac{1}{2}\big(|1,z\rangle + i\sqrt{2}|0,z\rangle - |-1,z\rangle\big)$ eigenvalue $\hbar$

$|0,y\rangle = |+\rangle = \frac{1}{\sqrt{2}}\big(|1,z\rangle + |-1,z\rangle\big)$ eigenvalue 0

$|-1,y\rangle = \frac{1}{\sqrt{2}}\big(|-\rangle - i|0,z\rangle\big) = \frac{1}{2}\big(|1,z\rangle - i\sqrt{2}|0,z\rangle - |-1,z\rangle\big)$ eigenvalue $-\hbar$

(c) $|\psi\rangle = |0,x\rangle = \frac{1}{\sqrt{2}}\left(|+1,z\rangle - |-1,z\rangle\right)$

Measure J_z: $P_{+1} = |\langle +1,z|0,x\rangle|^2 = \frac{1}{2}$

$P_0 = |\langle 0,z|0,x\rangle|^2 = 0$

$P_{-1} = |\langle -1,z|0,x\rangle|^2 = \frac{1}{2}$

Measure J_x: $P_{+1} = |\langle +1,x|0,x\rangle|^2 = 0$

$P_0 = |\langle 0,x|0,x\rangle|^2 = 1$

$P_{-1} = |\langle -1,x|0,x\rangle|^2 = 0$

Measure J_y: $P_{+1} = |\langle +1,y|0,x\rangle|^2 = \left|\frac{1}{2\sqrt{2}}(1+1)\right|^2 = \frac{1}{2}$

$P_0 = |\langle 0,y|0,x\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1-1)\right|^2 = 0$

$P_{-1} = |\langle -1,y|0,x\rangle|^2 = \left|\frac{1}{2\sqrt{2}}(1+1)\right|^2 = \frac{1}{2}$