7.3. Schrödinger equation: $i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$ $\quad H = \frac{1}{2}\hbar\omega\sigma_x$

$$\sigma_x = |+,z\rangle\langle -,z| + |-,z\rangle\langle +,z|$$

(a) Expand $|\psi(t)\rangle$ in the z basis:

$$|\psi(t)\rangle = c_+|+,z\rangle + c_-|-,z\rangle, \quad c_\pm = \langle \pm,z|\psi(t)\rangle$$

Project the Schrödinger equation onto the z basis:

$$i\hbar \underbrace{\langle +,z|\frac{d|\psi\rangle}{dt}} = \langle +,z|H|\psi(t)\rangle = \frac{1}{2}\hbar\omega \underbrace{\langle +,z|\sigma_x}_{\langle -,z|}|\psi(t)\rangle = \frac{1}{2}\hbar\omega c_-(t)$$

Because $\langle +,z|$ does not depend on t.

$$= \frac{d\langle +,z|\psi(t)\rangle}{dt} = \frac{dc_+}{dt} \implies \boxed{\frac{dc_+}{dt} = -\frac{i}{2}\omega c_-}$$

Similarly, $\boxed{\frac{dc_-}{dt} = -\frac{i}{2}\omega c_+}$

Solving: Differentiate the c_+ equation again.

$$\frac{d^2c_+}{dt} = -\frac{i}{2}\omega\frac{dc_-}{dt} = -\left(\frac{\omega}{2}\right)^2 c_+ \implies \ddot{c}_+ + (\omega/2)^2 c_+ = 0$$

This is one of the differential equations for which you must memorize the solution

Solution: $c_+(t) = c_+(0)\cos(\omega t/2) + \underbrace{\dot{c}_+(0)}_{-\frac{i}{2}\omega c_-(0)}\frac{2}{\omega}\sin(\omega t/2)$

$$c_+(t) = c_+(0)\cos(\omega t/2) - i\,c_-(0)\sin(\omega t/2)$$

$$c_-(t) = \frac{2i}{\omega}\dot{c}_+(t) = -i\,c_+(0)\sin(\omega t/2) + c_-(0)\cos(\omega t/2)$$

This is the general solution. For our initial condition, $|\psi(0)\rangle = |+,z\rangle$ or $c_+(0) = 1$, $c_-(0) = 0$, we have

$$\boxed{\begin{aligned} c_+(t) &= \cos(\omega t/2) \\ c_-(t) &= -i\sin(\omega t/2) \end{aligned}}$$

(b) Now let's do the same problem in the x basis:

$$|\pm,x\rangle = \frac{1}{\sqrt{2}}\left(|+,z\rangle \pm |-,z\rangle\right) \iff |\pm,z\rangle = \frac{1}{\sqrt{2}}\left(|+,x\rangle \pm |-,x\rangle\right).$$

σ_x is diagonal in this basis, with eigenvalues ± 1:

$$\sigma_x = |+,x\rangle\langle +,x| - |-,x\rangle\langle -,x| .$$

This is called the energy <u>eigenbasis</u> because $|\pm,x\rangle$ are the eigenstates of the Hamiltonian.

Expand $|\psi(t)\rangle$ in the x basis:

$$|\psi(t)\rangle = d_+|+,x\rangle + d_-|-,x\rangle, \quad d_\pm = \langle \pm,x|\psi(t)\rangle$$

Project the Schrödinger equation onto the x basis:

$$i\hbar \dot{d}_\pm = i\hbar\frac{d\langle\pm,x|\psi\rangle}{dt} = \langle\pm,x|H|\psi(t)\rangle = \frac{1}{2}\hbar\omega\underbrace{\langle\pm,x|\sigma_x}_{\pm\langle\pm,x|}|\psi(t)\rangle = \pm\frac{1}{2}\hbar\omega d_\pm$$

So $\dot{d}_\pm = \mp\frac{i}{2}\omega d_\pm \Rightarrow d_\pm(t) = d_\pm(0)\,e^{\mp i\omega t/2}$

This is the other differential equation for which you must memorize the solution.

This is general solution. It is always true that the probability amplitudes in the energy eigenbasis change in time according to a phase. For this reason, the energy eigenstates are called <u>stationary states</u>.

Now we need to put in the initial conditions. Generally, we have

$$d_\pm(t) = \langle\pm,x|\psi(t)\rangle = \frac{1}{\sqrt{2}}\Big(\underbrace{\langle +,z|\psi(t)\rangle}_{c_+(t)} \pm \underbrace{\langle -,z|\psi(t)\rangle}_{c_-(t)}\Big) = \frac{1}{\sqrt{2}}\left(c_+(t) \pm c_-(t)\right),$$

so $d_\pm(0) = \frac{1}{\sqrt{2}}$, and our solution is

$$\boxed{d_\pm(t) = \frac{1}{\sqrt{2}}\,e^{\mp i\omega t/2}}$$

(c) Generally,

$$c_\pm(t) = \langle \pm, z | \psi(t) \rangle = \frac{1}{\sqrt{2}}\Big(\underbrace{\langle +, x | \psi(t) \rangle}_{d_+(t)} \pm \underbrace{\langle -, x | \psi(t) \rangle}_{d_-(t)} \Big) = \frac{1}{\sqrt{2}}\big(d_+(t) \pm d_-(t)\big),$$

so for our initial conditions,

$$c_\pm(t) = \frac{1}{2}\left(e^{-i\omega t/2} \pm e^{+i\omega t/2}\right) \Longrightarrow \boxed{\begin{aligned} c_+(t) &= \cos(\omega t/2) \\ c_-(t) &= -i\sin(\omega t/2) \end{aligned}}$$

It works!

(d) Measurement of σ_z:

$$p_\pm = |\langle \pm, z | \psi(t) \rangle|^2 = |c_\pm(t)|^2 \Longrightarrow \boxed{\begin{aligned} p_+ &= \cos^2(\omega t/2) \\ p_- &= \sin^2(\omega t/2) \end{aligned}}$$

Measurement of σ_x:

$$q_\pm = |\langle \pm, x | \psi(t) \rangle|^2 = |d_\pm(t)|^2 \Longrightarrow \boxed{\begin{aligned} q_+ &= \frac{1}{2} \\ q_- &= \frac{1}{2} \end{aligned}}$$