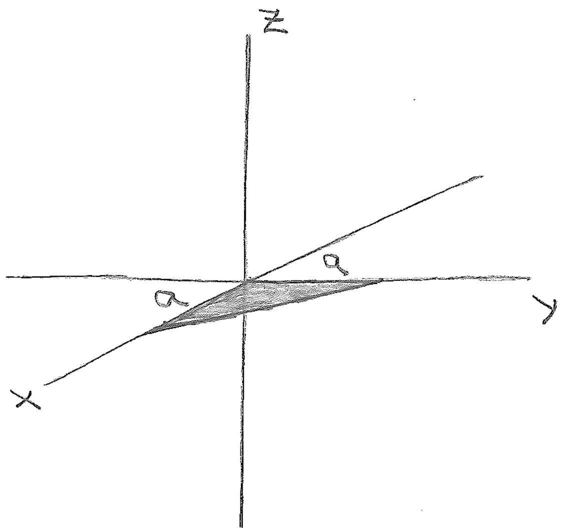
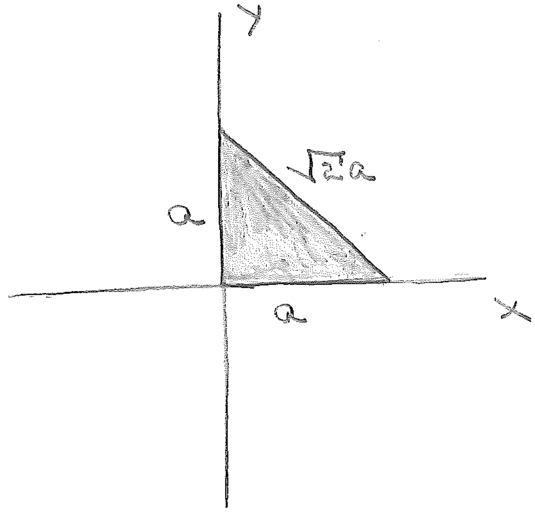
8.1





$$I_{jk} = \int dm\,(r^2\delta_{jk} - x_j x_k) \qquad M = \frac{1}{2}\sigma a^2$$

(a)
$$I_{xx} = \int dm\,(y^2+z^2) = \int dm\, y^2 = \frac{1}{6}Ma^2$$

$$I_{yy} = \int dm\,(x^2+z^2) = \int dm\, x^2 = \frac{1}{6}Ma^2$$

$$I_{zz} = \int dm\,(x^2+y^2) = \frac{1}{3}Ma^2$$

$$I_{xy} = -\int dm\, xy = -\frac{1}{12}Ma^2$$

$$I_{xz} = -\int dm\, xz = 0$$

$$I_{yz} = -\int dm\, yz = 0$$

Here are the necessary integrals:

$$\int dm\, y^2 = \sigma\int_0^a dy\, y^2 \underbrace{\int_0^{a-y} dx}_{a-y} = \sigma\underbrace{\int_0^a dy\,(ay^2 - y^3)}_{\frac{1}{3}a^4 - \frac{1}{4}a^4 = \frac{1}{12}a^4} = \frac{1}{12}\sigma a^4 = \frac{1}{6}Ma^2$$

$$\int dm\, x^2 = \frac{1}{6}Ma^2 \text{ by symmetry}$$

$$\int dm\, xy = \sigma \int_0^a dy\, y \underbrace{\int_0^{a-y} dx\, x}_{\frac{1}{2}(a-y)^2} = \frac{1}{2}\sigma \underbrace{\int_0^a dy\, y\,(a-y)^2}_{} = \frac{1}{12} M a^2$$

$$\int_0^a dy\, (a-y)\, y^2 = \frac{1}{12} a^4$$

$$\| I_{jk} \| = \frac{1}{3} M a^2 \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Eigenvalue equation: $\sum_k I_{jk} \omega_k = I \omega_j$ ← principal moment of inertia

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \lambda \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$\lambda = \dfrac{I}{\frac{1}{3} M a^2}$

We can see that $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 1.

Characteristic equation

$$0 = \det \begin{pmatrix} 1/2-\lambda & -1/4 & 0 \\ -1/4 & 1/2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = \left(\frac{1}{2}-\lambda\right)^2 (1-\lambda) + \frac{1}{4}\left(-\frac{1}{4}\right)(1-\lambda)$$

$$\Rightarrow \lambda = 1 \quad \text{or} \quad \left(\lambda - \frac{1}{2}\right)^2 - \frac{1}{16} = 0 \Leftrightarrow \lambda - \frac{1}{2} = \pm \frac{1}{4}$$

$$\lambda = 3/4 \quad \text{or} \quad 1/4$$

Principal moments:

1. $\lambda_1 = \frac{3}{4}$
2. $\lambda_2 = \frac{1}{4}$
3. $\lambda_3 = 1$

$$I_1 = \frac{1}{4} Ma^2$$

$$I_2 = \frac{1}{12} Ma^2$$

$$I_3 = \frac{1}{3} Ma^2$$

Principal axes:

1.
$$\left.\begin{array}{l} \frac{1}{2}\omega_1 - \frac{1}{4}\omega_2 = \frac{3}{4}\omega_1 \\ -\frac{1}{4}\omega_1 + \frac{1}{2}\omega_2 = \frac{3}{4}\omega_2 \end{array}\right] \Longrightarrow \omega_2 = -\omega_1$$

$$\omega_3 = \frac{3}{4}\omega_3 \Longrightarrow \omega_3 = 0$$

$\Longrightarrow$ Normalized eigenvector $\hat{x}' = \frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$

2.
$$\left.\begin{array}{l} \frac{1}{2}\omega_1 - \frac{1}{4}\omega_2 = \frac{1}{4}\omega_1 \\ -\frac{1}{4}\omega_1 + \frac{1}{2}\omega_2 = \frac{1}{4}\omega_2 \end{array}\right] \Longrightarrow \omega_2 = \omega_1$$

$$\omega_3 = \frac{1}{4}\omega_3 \Longrightarrow \omega_3 = 0$$

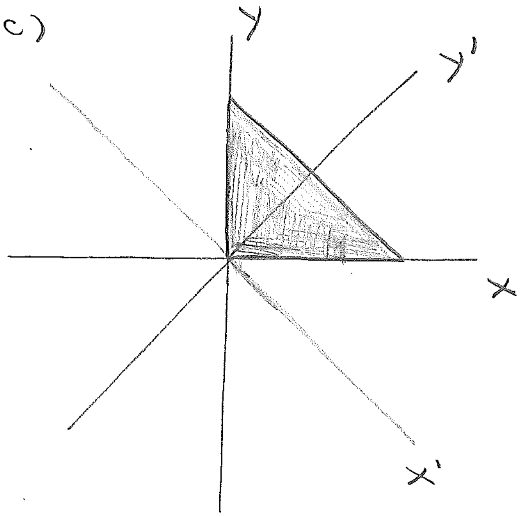
$\Longrightarrow$ Normalized eigenvector $\hat{y}' = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$

3.
$$\left.\begin{array}{l} \frac{1}{2}\omega_1 - \frac{1}{4}\omega_2 = \omega_1 \\ -\frac{1}{4}\omega_1 + \frac{1}{2}\omega_2 = \omega_2 \end{array}\right] \Longrightarrow \omega_1 = \omega_2 = 0$$

$$\omega_3 = \omega_3$$

$\Longrightarrow$ Normalized eigenvector $\hat{z}' = \hat{z}$

(c)



Calculate relative to the primed axes, but don't bother with the primes.

$$I_{xx} = \int dm\,(y^2+z^2) = \int dm\, y^2 = \frac{1}{4} M a^2 = I_1$$

$$I_{yy} = \int dm\,(x^2+z^2) = \int dm\, x^2 = \frac{1}{12} M a^2 = I_2$$

$$I_{zz} = \int dm\,(x^2+y^2) = \frac{1}{3} M a^2 = I_3$$

$$I_{xy} = -\int dm\, xy = 0$$ ⟵ reflection symmetry through y axis

$$I_{xz} = -\int dm\, xz = 0$$

$$I_{yz} = -\int dm\, yz = 0$$

The necessary integrals:

$$\int dm\, y^2 = \sigma \int_0^{a/\sqrt{2}} dy\, y^2 \int_{-y}^{+y} dx = 2\sigma \underbrace{\int_0^{a/\sqrt{2}} dy\, y^2 \underbrace{\int_0^{y} dx}_{y}}_{\frac{1}{4}\left(\frac{a}{\sqrt{2}}\right)^4} = \frac{1}{8}\sigma a^4 = \frac{1}{4} M a^2$$

$$\int dm\, x^2 = 2\sigma \int_0^{a/\sqrt{2}} dy \underbrace{\int_0^{y} dx\, x^2}_{\frac{1}{3}y^3} = \frac{1}{12} M a^2$$

$$\| I_{jk} \| = MR^2 \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

(d) Work in the primed coördinates, but drop the primes.

CM position

$$\vec{R} = \frac{1}{M}\int dm\, \vec{r} = \frac{1}{M}\int dm\, (x\hat{x} + y\hat{y} + z\hat{z})$$

$$= \frac{1}{M}\hat{y}\int dm\, y$$

$$= \frac{1}{M}\hat{y}\; 2\sigma \underbrace{\int_0^{a/\sqrt{2}} dy\, y \underbrace{\int_0^y dx}_{y}}_{\frac{1}{3}\left(\frac{a}{\sqrt{2}}\right)^3}$$

$$= \frac{1}{M}\hat{y}\,\frac{1}{3\sqrt{2}}\sigma a^3$$

$$= \frac{\sqrt{2}}{3} a\hat{y}$$

(f)

$$I_{jk} = I_{jk}^{(cm)} + \underbrace{M(R^2\delta_{jk} - X_j X_k)}_{\text{inertia tensor for all mass at cm}}$$

$$I_{jk}^{(cm)} = I_{jk} - M(R^2\delta_{jk} - X_j X_k)$$

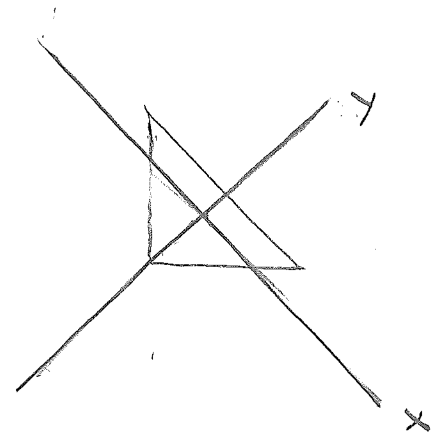
$$I_{xx}^{(cm)} = \frac{1}{4}Ma^2 - M\left(\frac{2}{9}a^2\right) = \frac{1}{36}Ma^2$$

$$I_{yy}^{(cm)} = \frac{1}{12}Ma^2$$

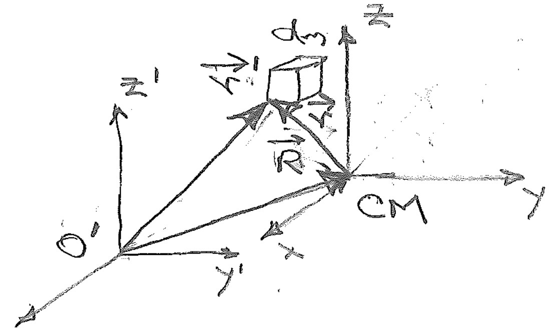
$$I_{zz}^{(cm)} = \frac{1}{3}Ma^2 - M\left(\frac{2}{9}a^2\right) = \frac{1}{9}Ma^2$$

Others zero

$$\| I_{jk}^{(cm)} \| = \frac{1}{3}Ma^2 \begin{pmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$



(e) Parallel-axis theorem



$$\vec{r}\,' = \vec{r} + \vec{R}, \quad x_j' = x_j + X_j$$

$$r'^2 = \vec{r}\,' \cdot \vec{r}\,' = r^2 + R^2 + 2\vec{r}\cdot\vec{R}$$

$$I_{jk}' = \int dm \left(r'^2 \delta_{jk} - x_j' x_k'\right)$$

$$= \int dm \left(r^2 \delta_{jk} - x_j x_k\right) + \underbrace{\int dm}_{M} \left(R^2 \delta_{jk} - X_j X_k\right)$$

$$+ \int dm \left(2\vec{r}\cdot\vec{R}\,\delta_{jk} - x_j X_k - x_k X_j\right)$$

$$= 2\delta_{jk}\vec{R}\cdot\underbrace{\int dm\,\vec{r}}_{0} - X_k \underbrace{\int dm\, x_j}_{0} - X_k \underbrace{\int dm\, x_k}_{0}$$

These are the CM positions in the CM coordinate system, where the CM is at the origin, so these integrals are zero

$$I'_{jk} = M(R^2 \delta_{jk} - X_j X_k) + I_{jk}^{(CM)}$$