8.2. (a) $\nabla\cdot\vec{S} = \frac{c}{4\pi}\nabla\cdot(\vec{E}\times\vec{B})$

$$= \frac{c}{4\pi}\Big(\vec{B}\cdot\underbrace{\nabla\times\vec{E}}_{-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}} - \vec{E}\cdot\underbrace{\nabla\times\vec{B}}_{\frac{1}{c}\frac{\partial\vec{E}}{\partial t}}\Big)$$

check the formula

← Maxwell equations

$$= -\frac{1}{4\pi}\Big(\underbrace{\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}}_{\frac{1}{2}\frac{\partial}{\partial t}(\vec{B}\cdot\vec{B})} + \underbrace{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}}_{\frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E})}\Big)$$

$$= -\frac{1}{8\pi}\frac{\partial}{\partial t}(E^2 + B^2)$$

$$= -\frac{\partial u}{\partial t}$$

(b) $\nabla\cdot\overleftrightarrow{T} = \hat{e}_j T_{jk,k}$

$$= \hat{e}_j \frac{1}{4\pi}\frac{\partial}{\partial x^k}\Big[-E_jE_k - B_jB_k + \frac{1}{2}\delta_{jk}(E_\ell E_\ell + B_\ell B_\ell)\Big]$$

$$= \hat{e}_j\frac{1}{4\pi}\Big[-E_{j,k}E_k - E_j\underbrace{E_{k,k}}_{\nabla\cdot\vec{E}=0} - B_{j,k}B_k - B_j\underbrace{B_{k,k}}_{\nabla\cdot\vec{B}=0}$$

$$+ \underbrace{\frac{1}{2}\delta_{jk}(2E_\ell E_{\ell,k} + 2B_\ell B_{\ell,k})}_{E_\ell E_{\ell,j} + B_\ell B_{\ell,j} = E_kE_{k,j} + B_kB_{k,j}}\Big]$$

$$= \frac{1}{4\pi}\Big[\hat{e}_jE_k\underbrace{(E_{k,j} - E_{j,k})} + \hat{e}_jB_k(\underbrace{B_{k,j} - B_{j,k}})\Big]$$

These are the components of $\nabla\times\vec{E}$ and $\nabla\times\vec{B}$, so we sort of think these have to do with $\vec{E}\times(\nabla\times\vec{E})$ and $\vec{B}\times(\nabla\times\vec{B})$.

Let's try it:

$$\vec{F}\times(\nabla\times\vec{F}) = \hat{e}_j\epsilon_{jk\ell}F_k\underbrace{(\nabla\times\vec{F})_\ell}_{\epsilon_{\ell mn}F_{n,m}} = \hat{e}_j\underbrace{\epsilon_{jk\ell}\epsilon_{\ell mn}}_{\epsilon_{\ell jk}\epsilon_{\ell mn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}}F_kF_{n,m} = \hat{e}_jF_k(F_{k,j} - F_{j,k})$$

So we were on the right track:

$$\nabla \cdot \overleftrightarrow{T} = \frac{1}{4\pi}\Big[\vec{E} \times \underbrace{(\nabla \times \vec{E})}_{-\frac{1}{c}\frac{\partial \vec{B}}{\partial t}} + \vec{B} \times \underbrace{(\nabla \times \vec{B})}_{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}\Big]$$

$\longleftarrow$ Maxwell equations

$$= -\frac{1}{4\pi c}\underbrace{\left(\vec{E} \times \frac{\partial \vec{B}}{\partial t} + \frac{\partial \vec{E}}{\partial t} \times \vec{B}\right)}_{\frac{\partial}{\partial t}(\vec{E} \times \vec{B})}$$

$$= -\frac{\partial}{\partial t}\left(\frac{1}{4\pi c}\vec{E} \times \vec{B}\right)$$

$$= -\frac{\partial(\vec{S}/c^2)}{\partial t}$$