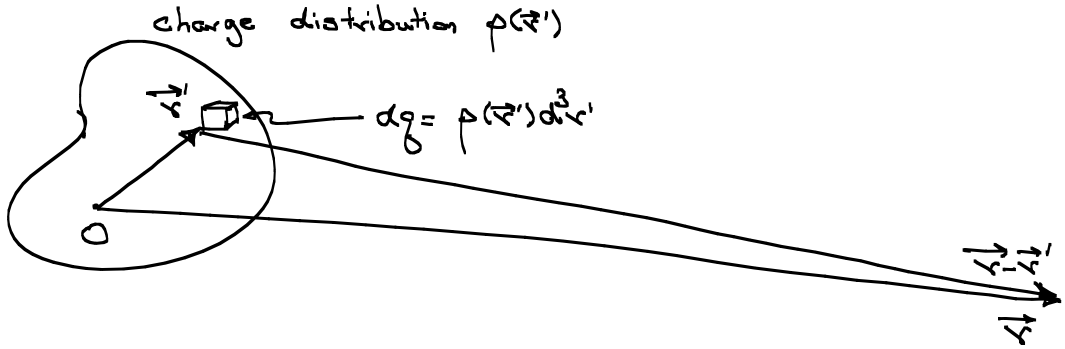
8.4



$$\Phi(\vec{r}) = \int \frac{dq}{|\vec{r}-\vec{r}\,'|}$$

(a)
$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \left[(\vec{r}-\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')\right]^{-1/2}$$
$$= \left(r^2 - 2\vec{r}\cdot\vec{r}\,' + r'^2\right)^{-1/2}$$
$$= r\left(1 - \underbrace{\frac{2\vec{r}\cdot\vec{r}\,'}{r^2}}_{\text{this term is } O(r'/r)} + \underbrace{\frac{r'^2}{r^2}}_{\text{this term is } O(r'/r)^2}\right)^{-1/2}$$

Expand this using

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{(-1/2)(-3/2)}{2!}x^2 + \ldots$$
$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \ldots$$

with $x = -\dfrac{2\vec{r}\cdot\vec{r}\,'}{r^2} + \dfrac{r'^2}{r^2}$. So

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\left[1 - \frac{1}{2}\left(-\underbrace{\frac{2\vec{r}\cdot\vec{r}\,'}{r^2}}_{O(r'/r)} + \underbrace{\frac{r'^2}{r^2}}_{O(r'/r)^2}\right) + \frac{3}{8}\underbrace{\left(-\frac{2\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2}\right)^2}_{\underbrace{\frac{4(\vec{r}\cdot\vec{r}\,')^2}{r^4}}_{O(r'^2/r^2)} + \ldots} + \ldots\right]$$

the other terms in the square are higher order than $(r'/r)^2$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\left[1 + \underbrace{\frac{\vec{r}\cdot\vec{r}\,'}{r^2}}_{O(r'/r)} + \underbrace{\frac{1}{2}\left(-\frac{r'^2}{r^2} + \frac{3(\vec{r}\cdot\vec{r}\,')^2}{r^4}\right)}_{O(r'/r)^2} + \underbrace{\ldots}_{O(r'/r)^3 \text{ and higher}}\right]$$

(b) $$\Phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$= \int d^3r' \rho(\vec{r}\,')\left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}\,'}{r^3} + \frac{1}{2}\left(-\frac{r'^2}{r^3} + \frac{3(\vec{r}\cdot\vec{r}\,')^2}{r^5}\right) + \dots\right]$$

$$= \frac{1}{r}\underbrace{\int d^3r' \rho(\vec{r}\,')}_{Q} + \frac{\vec{r}}{r^3}\cdot\underbrace{\int d^3r' \rho(\vec{r}\,')\vec{r}\,'}_{\vec{p}}$$

Write $(\vec{r}\cdot\vec{r}\,')^2$ as $(\vec{r}\cdot\vec{r}\,')^2 = x_j x_j' x_k x_k'$

$$+ \frac{1}{2}\left(-\frac{1}{r^3}\int d^3r' \rho(\vec{r}\,') r'^2 + \frac{x_j x_k}{r^5}\int d^3r' \rho(\vec{r}\,') 3x_j' x_k'\right) + \dots$$

Write $\frac{1}{r^3}$ as $\frac{r^2}{r^5} = \frac{x_j x_j}{r^5} = \frac{x_j x_k \delta_{jk}}{r^5}$

$$= \frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2}\sum_{j,k}\frac{x_j x_k}{r^5}\underbrace{\int d^3r' \rho(\vec{r}\,')\left(3x_j' x_k' - r'^2\delta_{jk}\right)}_{Q_{jk}} + \dots$$

$$\Phi(\vec{r}) = \frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \underbrace{\frac{1}{2}\sum_{jk}\frac{x_j x_k}{r^5}Q_{jk}}_{\text{quadrupole potential}} + \dots$$

monopole potential   dipole potential

$$= \frac{Q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{1}{2}\frac{\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{r^3} + \dots$$

This is the neatest form, because using $\hat{r}$ exposes the powers of $1/r$, but it is easier to take the gradient of the 1st form.

(c) Use $\nabla(f(r)) = \frac{df}{dr}\hat{r}$ and $\nabla x_j = \hat{e}_j$

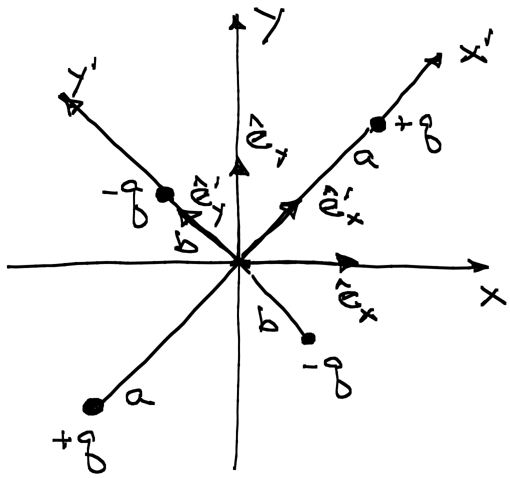
$$\nabla\Phi = Q\underbrace{\nabla\left(\frac{1}{r}\right)}_{-\frac{1}{r^2}\hat{r}} + p_j\underbrace{\nabla\left(\frac{x_j}{r^3}\right)}_{\frac{\hat{e}_j}{r^3} - \frac{3x_j\hat{r}}{r^4}} + \frac{1}{2}Q_{jk}\underbrace{\nabla\left(\frac{x_j x_k}{r^5}\right)}_{\frac{\hat{e}_j x_k + x_j\hat{e}_k}{r^5} - \frac{5x_j x_k}{r^6}\hat{r}} + \dots$$

$$= -\frac{Q}{r^3}\vec{r} + \frac{\vec{p} - 3(\vec{p}\cdot\hat{r})\hat{r}}{r^3} + \frac{1}{2}\left(\frac{\hat{e}_j Q_{jk} x_k + x_j Q_{jk}\hat{e}_k}{r^5} - \frac{5Q_{jk}x_j x_k\hat{r}}{r^6}\right) + \dots$$

Notice that $\hat{e}_j Q_{jk} x_k = \overleftrightarrow{Q}\cdot\vec{r} = \vec{r}\cdot\overleftrightarrow{Q} = x_k Q_{kj}\hat{e}_j$

$$= -\frac{Q}{r^3}\vec{r} + \frac{\vec{p} - 3(\vec{p}\cdot\hat{r})\hat{r}}{r^3} + \frac{1}{2}\frac{2\overleftrightarrow{Q}\cdot\hat{r} - 5(\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r})\hat{r}}{r^4} + \dots$$

$$\boxed{\vec{E} = -\nabla\Phi = \frac{Q}{r^2}\hat{r} + \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} + \frac{1}{2}\frac{5(\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r})\hat{r} - 2\overleftrightarrow{Q}\cdot\hat{r}}{r^4} + \dots}$$



Total charge is zero

Dipole moment is like the center of mass and thus is zero.

(d) $Q_{jk} = \sum_\alpha q_\alpha\left[3x_j^\alpha x_k^\alpha - \delta_{jk}(r^\alpha)^2\right]$ α labels which charge

$$\sum_\alpha q_\alpha (r^\alpha)^2 = 2qa^2 - 2qb^2 = 2q(a^2 - b^2)$$

$$\sum_\alpha q_\alpha x^\alpha x^\alpha = 2q\frac{a^2}{2} - 2q\frac{b^2}{2} = \sum_\alpha q_\alpha y^\alpha y^\alpha$$
$$= q(a^2 - b^2)$$

$$\sum_\alpha q_\alpha x^\alpha y^\alpha = 2q\frac{a^2}{2} - 2q\left(-\frac{b^2}{2}\right) = q(a^2+b^2)$$

$$Q_{xx} = 3q(a^2-b^2) - 2q(a^2-b^2) = q(a^2-b^2)$$
$$Q_{yy} = q(a^2-b^2)$$
$$Q_{xy} = Q_{yx} = 3q(a^2+b^2)$$
$$Q_{zz} = -2q(a^2-b^2)$$
$$Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$$

Symmetric and traceless

(e) The matrix for $\overleftrightarrow{Q}$ is

$$\begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{pmatrix} = q \begin{pmatrix} a^2-b^2 & 3(a^2+b^2) & 0 \\ 3(a^2+b^2) & a^2-b^2 & 0 \\ 0 & 0 & -2(a^2-b^2) \end{pmatrix}$$

It is obvious that $\hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $-2q(a^2-b^2)$. Moreover, the symmetry of the charge distribution means that the other two eigenvectors are

$$\hat{e}'_x = \frac{1}{\sqrt{2}}(\hat{e}_x + \hat{e}_y) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{e}'_y = \frac{1}{\sqrt{2}}(-\hat{e}_x + \hat{e}_y) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

We get the eigenvalues from

$$q \begin{pmatrix} a^2-b^2 & 3(a^2+b^2) & 0 \\ 3(a^2+b^2) & a^2-b^2 & 0 \\ 0 & 0 & -2(a^2-b^2) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \underbrace{2q(2a^2+b^2)}_{\substack{\text{eigenvalue} \\ \text{for } \hat{e}'_x}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$q \begin{pmatrix} a^2-b^2 & 3(a^2+b^2) & 0 \\ 3(a^2+b^2) & a^2-b^2 & 0 \\ 0 & 0 & -2(a^2-b^2) \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \underbrace{-2q(a^2+2b^2)}_{\substack{\text{eigenvalue} \\ \text{for } \hat{e}'_y}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

(f) Now, using the primed coördinates,

$$Q'_{jk} = \sum_\alpha q^\alpha \left[3x_j^{\alpha'} x_k^{\alpha'} - \delta_{jk}(r^{\alpha'})^2\right]$$

$$\sum_\alpha q^\alpha (r^{\alpha'})^2 = 2q(a^2-b^2) \quad \text{as before}$$

$$\sum_\alpha q^\alpha x^{\alpha'} x^{\alpha'} = 2q a^2$$ only the positive charges are on the x' axis

$$\sum_\alpha q^\alpha y^{\alpha'} y^{\alpha'} = -2q b^2$$ only the negative charges are on the y' axis

$$\sum_\alpha q^\alpha x^{\alpha'} y^{\alpha'} = 0$$ ⟵ no charge has both nonzero x' and y' coördinates

So

$$\boxed{\begin{aligned} Q'_{xx} &= 6qa^2 - 2q(a^2-b^2) = 2q(2a^2+b^2) \\ Q'_{yy} &= -6qb^2 - 2q(a^2-b^2) = -2q(a^2+2b^2) \\ Q'_{zz} &= -2q(a^2-b^2) \\ &\text{All off-diagonals are zero} \end{aligned}}$$

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