9.1 Boas Sec. 7.5

We will use the Boas conventions for a periodic function with period $L = 2\pi$.

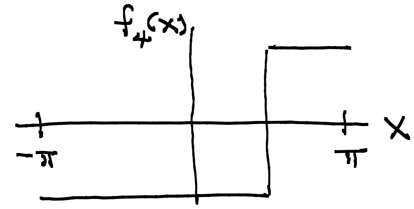
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n})$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, f(x)\, e^{-inx}$$

Since all the functions are real, we have $c_{-n} = c_n^*$, which implies $a_n = 2\,\mathrm{Re}\,c_n$ and $b_n = -2\,\mathrm{Im}\,c_n$.

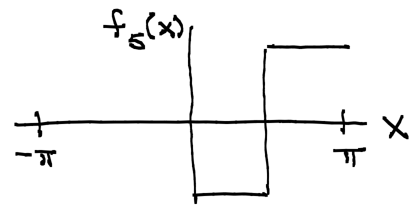
Problem 4. $f_4(x) = \begin{cases} -1, & -\pi < x < \pi/2 \\ 1, & \pi/2 < x < \pi \end{cases}$



Use Problem 3: $f_4(x) = 2f_3(x) - 1$

$$= -\frac{1}{2} - \frac{2}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \cdots\right.$$
$$\left. + \frac{2}{\pi}\left(\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \frac{2\sin 6x}{6} + \cdots\right)\right)$$

Problem 5. $f_5(x) = \begin{cases} 0, & -\pi < x < 0 \\ -1, & 0 < x < \pi/2 \\ 1, & \pi/2 < x < \pi \end{cases}$

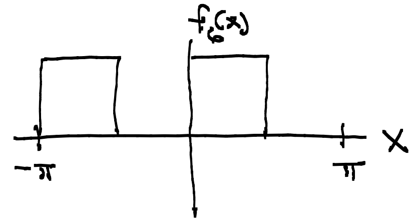


Use Problems 1 and 3.

$$f_5(x) = f_1(x) - 1 + 2f_3(x) = f_1(x) + f_4(x)$$

$$= -\frac{1}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right)$$
$$+ \frac{1}{2} - \frac{2}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \cdots\right)$$
$$+ \frac{2}{\pi}\left(\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \frac{2\sin 6x}{6} + \cdots\right)$$

$$= -\frac{2}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \cdots\right)$$
$$- \frac{2}{\pi}\left(\sin 2x + \frac{\sin 6x}{3} + \cdots\right)$$



Problem 6. $f_6(x) = \begin{cases} 1, & -\pi < x < -\frac{\pi}{2} \text{ and } 0 < x < \frac{\pi}{2} \\ 0, & -\frac{\pi}{2} < x < 0 \text{ and } \frac{\pi}{2} < x < \pi \end{cases}$

Use $f_6(x) = 1 + \underbrace{\frac{1}{2}\left[f_5(x) + f_5(x+\pi)\right]}$

$$= -\frac{1}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots\right) - \frac{1}{\pi}\left(\sin 2x + \frac{\sin 6x}{3} + \dots\right)$$
$$+ \frac{1}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots\right) - \frac{1}{\pi}\left(\sin 2x + \frac{\sin 6x}{3} + \dots\right)$$
$$= 1 - \frac{2}{\pi}\left(\sin 2x + \frac{\sin 6x}{3} + \dots\right)$$

Problem 11. $f_{11}(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, f(x)\, e^{-inx}$$
$$= \frac{1}{2\pi}\int_0^{\pi} dx\, \sin x\, e^{-inx}$$
$$= \frac{1}{2\pi}\frac{1}{2i}\int_0^{\pi} dx\left(e^{ix} - e^{-ix}\right)e^{-inx}$$
$$= \frac{1}{4\pi i}\underbrace{\int_0^{\pi} dx\, e^{-i(n-1)x}}_{\frac{i}{n-1}\left(e^{-i(n-1)\pi} - 1\right)} - \frac{1}{4\pi i}\underbrace{\int_0^{\pi} dx\, e^{-i(n+1)x}}_{\frac{i}{n+1}\left(e^{-i(n+1)\pi} - 1\right)}$$
$$= \frac{1}{4\pi}\left(\frac{(-1)^{n-1} - 1}{n-1} - \frac{(-1)^{n+1} - 1}{n+1}\right)$$

We have to do $n = \pm 1$ separately, either directly or as limits:

$C_1 = -i/4 = -C_{-1}$

$$C_n = \begin{cases} \mp\frac{i}{4}, & n = \pm 1 \\ 0, & n \text{ odd}, n \neq \pm 1 \\ -\frac{1}{\pi}\frac{1}{n^2-1}, & n \text{ even} \end{cases}$$

$$a_n = 2\,\mathrm{Re}\, C_n = \begin{cases} 0, & n \text{ odd} \\ -\frac{2}{\pi}\frac{1}{n^2-1}, & n \text{ even} \end{cases} \qquad b_n = -2\,\mathrm{Im}\, C_n = \begin{cases} \frac{1}{2}, & n = +1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{11}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{\cos nx}{n^2-1}$$