Phys 366

Lecture 1

Complex numbers. I

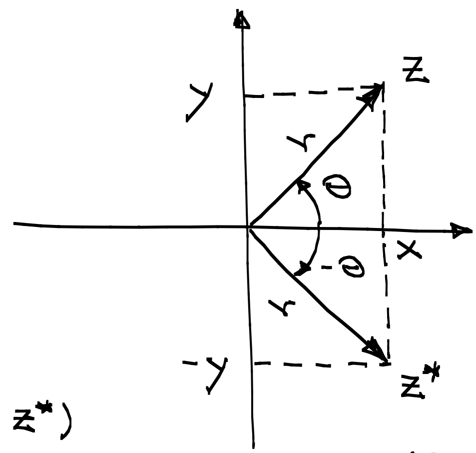
Complex numbers

Why? Find complete set of solutions for all polynomials with real (and then complex) coefficients.

Unit real number: 1 $\quad 1^2 = 1$

Unit complex number: i $\quad i^2 = -1$

Complex numbers as vectors



$$z = x + iy = re^{i\theta}$$

modulus $|z|$ (magnitude)

phase $\theta = \arg z$

real part

imaginary part

$$y = \mathrm{Im}(z) = -\frac{i}{2}(z - z^*)$$

$$x = \mathrm{Re}(z) = \frac{1}{2}(z + z^*)$$

$x = r\cos\theta \qquad r = \sqrt{x^2+y^2} = |z|$

$y = r\sin\theta \qquad \tan\theta = y/x$

Complex conjugate: $z^* = x - iy = re^{-i\theta}$

sometimes written $\bar{z}$, as in Boas

Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ ← vector addition

Multiplication by reals: $az = ax + iay$ ← scalar multiplication

But there is also a goofy rule for multiplying these vectors:

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + \underbrace{i^2}_{=-1} y_1y_2 + ix_1y_2 + iy_1x_2$$

$$\boxed{z_1 z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)}$$

$$\Rightarrow zz^* = x^2 + y^2 = r^2 = |z|^2$$

Euler's relation: $e^{i\theta} = \cos\theta + i\sin\theta \implies$ Neatest formula in all of math $e^{i\pi} = -1$

Complex numbers are an efficient way of doing trig.
So where does Euler's relation come from?

Definitions:

$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \ldots = \sum_{n \text{ even}} \frac{(-)^{n/2}}{n!} z^n = \sum_{n \text{ even}} \frac{i^n}{n!} z^n$$

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \ldots = \sum_{n \text{ odd}} \frac{(-)^{(n-1)/2}}{n!} z^n = -i \sum_{n \text{ odd}} \frac{i^n}{n!} z^n$$

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

$$\implies \cos z + i\sin z = \sum_{n=0}^{\infty} \frac{1}{n!}(iz)^n = e^{iz}$$

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \,?$$

$$e^{z_1}e^{z_2} = \left(\sum_{k=0}^{\infty} \frac{1}{k!} z_1^k\right)\left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} z_2^\ell\right)$$

$$= \sum_{k,\ell=0}^{\infty} \frac{1}{k!\,\ell!} z_1^k z_2^\ell$$

$N = k + \ell = 0, 1, 2, 3, \ldots$
$n = k = 0, 1, \ldots, N$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \underbrace{\sum_{n=0}^{N} \frac{N!}{n!(N-n)!} z_1^n z_2^{N-n}}_{\text{binomial sum } (z_1+z_2)^N}$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} (z_1+z_2)^N$$

$$= e^{z_1+z_2}$$

Multiplication:

$$z_1 z_2 = (x_1+iy_1)(x_2+iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

add the angles

Inverse: ① $z^{-1} = \frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{z^*}{r^2} = \frac{x-iy}{r^2}$

② $z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}$

Division: ① $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$ ② $\dfrac{z_1}{z_2} = \dfrac{z_1 z_2^*}{|z_2|^2}$

Trig identities:

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$$

$$\Longrightarrow \begin{cases} \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \\ \sin 2\theta = 2\cos\theta\sin\theta \end{cases}$$

$$\Longrightarrow \cos^2(\theta/2) = \tfrac{1}{2}(1+\cos\theta), \quad \sin^2(\theta/2) = \tfrac{1}{2}(1-\cos\theta)$$

Logarithm (natural):

$$\ln z = \ln(re^{i\theta}) = \ln r + \underset{\substack{\uparrow \\ \text{multivalued}}}{i\theta} = \ln\sqrt{x^2+y^2} + i\underset{\substack{\uparrow \\ \text{arctan}}}{\tan^{-1}}(y/x)$$

Derivatives:

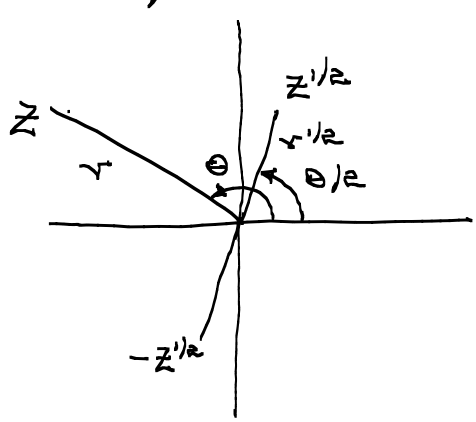
Chain rule: $\dfrac{d}{dz}\left(e^{f(z)}\right) = \dfrac{de^f}{df}\dfrac{df}{dz} = \dfrac{df}{dz} e^{f(z)}$

OR $\dfrac{d}{dz}\left(e^{f(z)}\right) = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\frac{d}{dz}[f(z)]^k}_{k\frac{df}{dz}[f(z)]^{k-1}} = \frac{df}{dz} \underbrace{\sum_{k=1}^{\infty} \frac{1}{(k-1)!}[f(z)]^{k-1}}_{e^{f(z)}} = \frac{df}{dz} e^{f(z)}$

$$\frac{d}{dx}(e^{ikx}) = ike^{ikx} \qquad \nabla\left(e^{i\vec{k}\cdot\vec{r}}\right) = i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$$

Taking a square root:

$z = x+iy = re^{i\theta}, \quad 0 \le \theta < 2\pi$



$i^{1/2} = e^{i\pi/4}$

$(-i)^{1/2} = e^{i3\pi/4}$

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2} = \sqrt{r}\,\big(\underbrace{\cos(\theta/2)}_{=\pm\frac{1}{\sqrt{2}}(1+\cos\theta)^{1/2}} + i\,\underbrace{\sin(\theta/2)}_{\frac{1}{\sqrt{2}}(1-\cos\theta)^{1/2}}\big)$$

upper sign: $0 \le \theta \le \pi$ $(y \ge 0)$

lower sign: $\pi \le \theta \le 2\pi$ $(y \le 0)$

There are two roots, $z^{1/2}$ and $-z^{1/2}$. In a physics problem, the physics will tell you which to use or both.

$$= \pm \frac{1}{\sqrt{2}}\left(r + r\cos\theta\right)^{1/2} + \frac{i}{\sqrt{2}}\left(r - r\cos\theta\right)^{1/2}$$

$$= \pm \frac{1}{\sqrt{2}}\left(\sqrt{x^2+y^2} + x\right)^{1/2} + \frac{i}{\sqrt{2}}\left(\sqrt{x^2+y^2} - x\right)^{1/2}$$

$$\boxed{z^{1/2} = \sqrt{r}\,e^{i\theta/2} = \pm \frac{1}{\sqrt{2}}\left(\sqrt{x^2+y^2} + x\right)^{1/2} + \frac{i}{\sqrt{2}}\left(\sqrt{x^2+y^2} - x\right)^{1/2}}$$

upper sign: $0 \le \theta \le \pi \iff y \ge 0$

lower sign: $\pi \le \theta \le 2\pi \iff y \le 0$

Let's check: $z^{1/2} z^{1/2} = \frac{1}{2}\left(\sqrt{x^2+y^2} + x\right) - \frac{1}{2}\left(\sqrt{x^2+y^2} - x\right)$

$$\pm 2\frac{i}{2}\underbrace{\left(\sqrt{x^2+y^2}+x\right)^{1/2}\left(\sqrt{x^2+y^2}-x\right)^{1/2}}_{(x^2+y^2-x^2)^{1/2} = \sqrt{y^2} = |y|}$$

$$= x \pm i|y|$$

$$= x + iy = z$$

nth roots of unity: $1^{1/n} = (e^{2\pi i k})^{1/n} = e^{2\pi i k/n}$, $k = 0, 1, \dots, n-1$

Solving all polynomials with real coefficients: We take it as given that all polynomials can be factored into a product of linear polynomials (real roots) and quadratic polynomials. So all we need to do is to solve the quadratic equation

$$z^2 + bz + c = 0, \quad z = x + iy \implies \begin{aligned} x^2 - y^2 + bx + c &= 0 \\ 2xy + by &= 0 \end{aligned}$$

$y \ne 0$, so $x = -b/2$

$$\therefore \quad y^2 = c - b^2/4 \implies y = \pm \frac{1}{2}\sqrt{4c - b^2}$$