Phys 366

Lecture 16

Tensors. General considerations

Rank-2 tensor

① A rank-2 tensor is a linear map from vectors to vectors

Stress tensor: $\overleftrightarrow{T}\cdot\hat{n}\,da = \begin{pmatrix}\text{force } -\hat{n} \text{ side exerts} \\ \text{on the } +\hat{n} \text{ side of } da\end{pmatrix}$

Inertia tensor: $\overleftrightarrow{I}\cdot\vec{\omega} = \vec{L}$

Abstract: $$\overleftrightarrow{T}\cdot\vec{A} = \sum_k (\overleftrightarrow{T}\cdot\hat{e}_k)(\hat{e}_k\cdot\vec{A})$$

$$= \sum_{j,k} \hat{e}_j \underbrace{(\hat{e}_j\cdot\overleftrightarrow{T}\cdot\hat{e}_k)}_{T_{jk}} \underbrace{(\hat{e}_k\cdot\vec{A})}_{A_k}$$

Bra-ket version

$$T|A\rangle = \sum_{j,k} |e_j\rangle \underbrace{\langle e_j|T|e_k\rangle}_{T_{jk}} \underbrace{\langle e_k|A\rangle}_{A_k}$$

$$= \sum_{j,k} \hat{e}_j T_{jk} A_k$$

Matrix representation

$$\longleftrightarrow \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Matrix representation

$$\overleftrightarrow{T} = \sum_{j,k} T_{jk}\, \hat{e}_j \otimes \hat{e}_k \qquad \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

Bra-ket version

$$T = \sum_{j,k} |e_j\rangle \underbrace{\langle e_j|T|e_k\rangle}_{T_{jk}} \langle e_k|$$

$$\vec{B}\cdot\overleftrightarrow{T}\cdot\vec{A} = \sum_{j,k} B_j T_{jk} A_k$$

Bra-ket version

$$\langle B|T|A\rangle = \sum_{j,k} \underbrace{\langle B|e_j\rangle}_{B_j^*} \underbrace{\langle e_j|T|e_k\rangle}_{T_{jk}} \underbrace{\langle e_k|A\rangle}_{A_k}$$

Matrix representation

$$\longleftrightarrow (B_1\ B_2\ B_3)\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

② A rank-2 tensor is a bilinear map from two vectors to a scalar.

Example: $K = \frac{1}{2}\vec{L}\cdot\vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{L}\cdot\vec{\omega} = \frac{1}{2}\sum_{j,k} L_{jk}\,\omega_j\omega_k$

Passive orthogonal transformations:

Vector: $A'_j = \hat{e}'_j\cdot\vec{A} = \sum_k \underbrace{\hat{e}'_j\cdot\hat{e}_k}_{M_{jk}} A_k = \sum_{j,k} M_{jk} A_k$

$M_{jk} \longleftarrow$ orthogonal matrix

$$\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Bra-ket version

$A'_j = \langle e'_j|A\rangle = \sum_k \underbrace{\langle e'_j|e_k\rangle}_{\substack{U_{jk} \\ \text{unitary} \\ \text{matrix}}}\underbrace{\langle e_k|A\rangle}_{A_k} = \sum_k U_{jk}A_k$

2-tensor: $T'_{jk} = \hat{e}'_j\cdot\overleftrightarrow{T}\cdot\hat{e}'_k = \sum_{l,m}\underbrace{(\hat{e}'_j\cdot\hat{e}_l)}_{M_{jl}}\underbrace{\hat{e}_l\cdot\overleftrightarrow{T}\cdot\hat{e}_m}_{T_{lm}}\underbrace{(\hat{e}_m\cdot\hat{e}'_k)}_{M_{km}=M^T_{mk}} = \sum_{l,m} M_{jl}T_{lm}M^T_{mk}$

$$\begin{pmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}\begin{pmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{32} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}$$

Bra-ket version

$\langle e'_j|T|e'_k\rangle = \sum_{l,m}\underbrace{\langle e'_j|e_l\rangle}_{U_{jl}}\langle e_l|T|e_m\rangle\underbrace{\langle e_m|e'_k\rangle}_{U^*_{km}=(U^\dagger)_{mk}}$

A rank-l tensor is a multilinear map from l vectors to a scalar.

$l=3$: $\mathbf{T}(\vec{A},\vec{B},\vec{C}) = \mathbf{T}(A_j\hat{e}_j, B_k\hat{e}_k, C_l\hat{e}_l)$

$= A_jB_kC_l\underbrace{\mathbf{T}(\hat{e}_j,\hat{e}_k,\hat{e}_l)}_{T_{jkl}}$

$T_{jkl} \longleftarrow$ components of $\mathbf{T}$

$= T_{jkl}A_jB_kC_l$

$\mathbf{T} = T_{jkl}\,\hat{e}_j\otimes\hat{e}_k\otimes\hat{e}_l \longleftarrow$ how does this work?

Passive orthogonal transformations:

$T'_{jkl} = \mathbf{T}(\hat{e}'_j, \hat{e}'_k, \hat{e}'_l)$

Passive orthogonal transformations:

$$T'_{jkl} = \mathbf{T}(\hat{e}'_j, \hat{e}'_k, \hat{e}'_l) = T_{mnp}\underbrace{(\hat{e}'_j \cdot e_m)}_{M_{jm}}\underbrace{(\hat{e}'_k \cdot \hat{e}_n)}_{M_{kn}}\underbrace{(\hat{e}'_l \cdot \hat{e}_p)}_{M_{lp}} = M_{jm} M_{kn} M_{lp} T_{mnp}$$

Each index transforms like a vector.

Contraction (tracing)

$$\mathbf{T} = T_{jkl}\, \hat{e}_j \otimes \hat{e}_k \otimes \hat{e}_l$$

Contraction on 1st and 3rd slots

$$\mathbf{S} = T_{jkl}\, \hat{e}_k \underbrace{(\hat{e}_j \cdot \hat{e}_l)}_{\delta_{jl}} = T_{jkj}\, \hat{e}_k \longleftarrow$$

$\mathbf{S}$ is a vector. Contraction reduces the rank by two.

$$\mathbf{S}(\vec{A}) = \vec{S} \cdot \vec{A} = T_{jkj}\, \hat{e}_k \cdot \vec{A} = T_{jkj} A_k$$

$$S'_k = T'_{jkj} = M_{jl} M_{km} M_{jn} T_{lmn} = M_{km} T_{lml} = M_{km} S_m$$

$$M^T_{lj} M_{jn} = (M^T M)_{ln} = \delta_{ln}$$

Decomposition of two tensors: start with T_{jk}

$$S_{jk} = \frac{1}{2}(T_{jk} + T_{kj}) = T_{(jk)} \longleftarrow \text{symmetric part}$$

$$A_{jk} = \frac{1}{2}(T_{jk} - T_{kj}) = T_{[jk]} \longleftarrow \text{antisymmetric part}$$

$$T = T_{jj} = T_{11} + T_{22} + T_{33} = S_{jj} = S \longleftarrow \text{trace, a scalar}$$

$$S^T_{jk} = S_{jk} - \frac{1}{3} S\, \delta_{jk} \longleftarrow \text{symmetric, trace-free part}$$

Symmetric part: find the eigenvectors and eigenvalues

$$\overleftrightarrow{S} \cdot \hat{\lambda}_j = \lambda_j \hat{\lambda}_j$$

Bra-ket version
$S|\lambda\rangle = \lambda|\lambda\rangle$

Characteristic equation: $\det(\overleftrightarrow{S} - \lambda_j \overleftrightarrow{I}) = 0$

A real symmetric matrix has 3 real eigenvalues λ_j and and corresponding orthonormal eigenvectors $\hat{\lambda}_j$.

What is the matrix representation of $\overleftrightarrow{S}$ in its eigenbasis?

$$\hat{\lambda}_j \cdot \underbrace{\overleftrightarrow{S} \cdot \hat{\lambda}_k}_{\lambda_k \hat{\lambda}_k} = \lambda_k \underbrace{\hat{\lambda}_j \cdot \hat{\lambda}_k}_{\delta_{jk}} = \lambda_j \delta_{jk}$$

$\overleftrightarrow{S}$ is diagonal in its eigenbasis.

$$\overleftrightarrow{S} = \sum_j \lambda_j \hat{\lambda}_j \otimes \hat{\lambda}_j$$

Bra-ket version

$\langle \lambda_j | S | \lambda_k \rangle = \lambda_j \delta_{jk}$

$S = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_k|$

A symmetric tensor is characterized by 3 eigendirections and the associated eigenvalues, i.e., an ellipsoid if the eigenvalues are positive. If the tensor is traceless, the 3 eigenvalues add to zero.

How is this diagonal representation related to the original representation?

$$\lambda_j \delta_{jk} = \hat{\lambda}_j \cdot \overleftrightarrow{S} \cdot \hat{\lambda}_k = \underbrace{\hat{\lambda}_j \cdot \hat{e}_\ell}_{M_{j\ell}} S_{\ell m} \underbrace{\hat{e}_m \cdot \hat{\lambda}_k}_{M_{km} = M^T_{mk}} = M_{j\ell} S_{\ell m} M^T_{mk}$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}$$

A real symmetric is diagonalized by an orthogonal transformation.

Going the other way:

$$S_{jk} = \hat{e}_j \cdot \overleftrightarrow{S} \cdot \hat{e}_k = \sum_\ell \underbrace{\hat{e}_j \cdot \hat{\lambda}_\ell}_{M_{\ell j} = M^T_{j\ell}} \lambda_\ell \underbrace{\hat{\lambda}_\ell \cdot \hat{e}_k}_{M_{\ell k}} = \sum_\ell M^T_{j\ell} \lambda_\ell M_{\ell k}$$

Antisymmetric part: $A_{jk} \longleftrightarrow \begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix}$

$A_{jk} = \epsilon_{jk\ell} A_\ell \qquad A_\ell = \frac{1}{2} \epsilon_{\ell jk} A_{jk}$

These are the components of a vector.

Example: $\vec{A} = \vec{B} \times \vec{C}$ $\qquad A_j = \epsilon_{jkl} B_k C_l = \frac{1}{2} \epsilon_{jkl} \underbrace{(B_k C_l - C_l B_k)}_{A_{jk}}$

Does it transform like a vector?

$$\underbrace{M_{jk}}_{\hat{e}_j' \cdot \hat{e}_k} A_k = \frac{1}{2} M_{jk} \epsilon_{klm} \underbrace{A_{lm}} = \frac{1}{2} \underbrace{\epsilon_{klm} M_{jk} M_{nl} M_{pm}}_{(\det M)\, \epsilon_{jnp}} A'_{np} = (\det M) A'_j$$

$$= \hat{e}_l \cdot \overleftrightarrow{A} \cdot \hat{e}_m = \hat{e}_l \cdot \hat{e}'_n A'_{np} \hat{e}'_p \cdot \hat{e}_m = M_{nl} M_{pm} A'_{np}$$

$A'_j = (\det M) M_{jk} A_k$

$\vec{A}$ transforms like a vector under rotations, but changes sign relative to a vector under inversions and reflections.

$\vec{A}$ is a <u>pseudo</u>vector.

A real antisymmetric 2-tensor is a pseudovector.