Phys 366

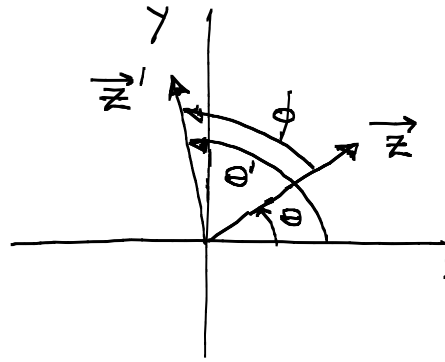
Lecture 2

Complex numbers. II

Complex numbers.

Why? They capture the geometry of two-dimensional space.

2-d vectors $\vec{z} = x\hat{i} + y\hat{j} = \begin{pmatrix} x \\ y \end{pmatrix}$



Rotate $\vec{z}$ by angle ϕ to get $\vec{z}'$.
This is an <u>active</u> <u>transformation</u>.

A rotation is a <u>linear</u> transformation, so it is described by a matrix, in this case a 2x2 real matrix M, that transforms the Cartesian components of vectors

$r' = r$
$\theta' = \theta + \phi$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}, \quad R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Try to remember. The only question is where to put the minus sign.

orthogonal matrix
$RR^T = I$

$$x' = x\cos\phi - y\sin\phi$$
$$y' = x\sin\phi + y\cos\phi$$

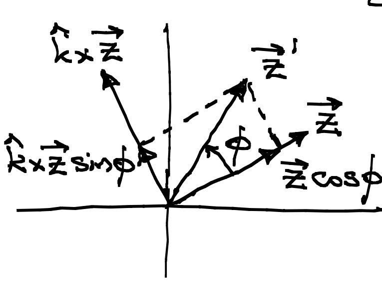
<u>or</u> $\vec{z}' = R\vec{z}$

3rd basis vector

Another representation: $R = \cos\phi\, I + \sin\phi\, \hat{k}\times$

vector cross product

$$\vec{z}' = R\vec{z} = \vec{z}\cos\phi + \hat{k}\times\vec{z}\sin\phi$$



$\hat{k}\times$ is a machine for rotating vectors by 90°.

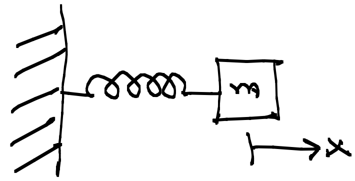
$$\vec{z}' = R\vec{z} = (\cos\phi\, I + \sin\phi\, \hat{k}\times)(x\hat{i} + y\hat{j})$$

This sign change is the point

$$= x\cos\phi\,\hat{i} + y\cos\phi\,\hat{j} + x\sin\phi\,\hat{j} + y\sin\phi\,(-\hat{i})$$
$$= (x\cos\phi - y\sin\phi)\hat{i} + (x\sin\phi + y\cos\phi)\hat{j}$$

This 2d geometry with rotations is the entire story in <u>linear</u> systems (linear mechanical systems, E&M, quantum mechanics).

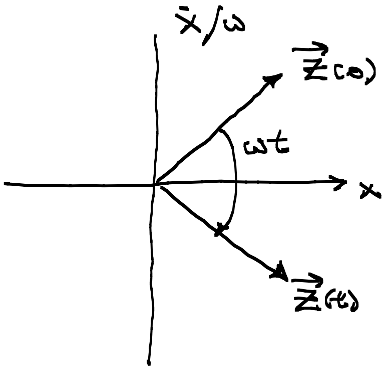
SHO:



$$F = -kx$$

$$m\ddot{x} = -kx \iff \ddot{x} + \omega^2 x = 0, \quad \omega^2 = \frac{k}{m}$$

Solution:

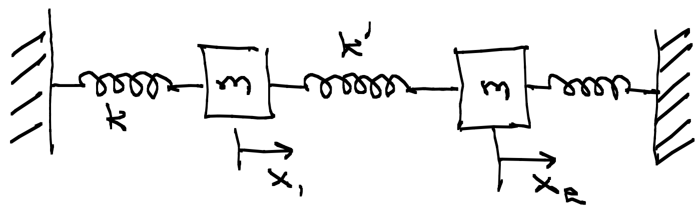


$$x(t) = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t$$

$$\frac{\dot{x}(t)}{\omega} = -x_0 \sin\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t$$

$$\vec{Z}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t)/\omega \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \underbrace{\begin{pmatrix} x_0 \\ \dot{x}_0/\omega \end{pmatrix}}_{\vec{Z}(0)} = \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \vec{Z}(0)$$

Coupled linear systems:



$$m\ddot{x}_1 = F_1 = -kx_1 + k'(x_2 - x_1)$$

$$m\ddot{x}_2 = F_2 = -kx_2 - k'(x_2 - x_1)$$

$$F_1 = -kx_1 + k'(x_2 - x_1)$$

$$F_2 = -k'(x_2 - x_1) - kx_2$$

$$\left.\begin{aligned} q_+ &= x_1 + x_2 \\ q_- &= x_1 - x_2 \end{aligned}\right\} \text{normal-mode coördinates}$$

$$\begin{aligned} m\ddot{q}_+ &= -kq_+ \\ m\ddot{q}_- &= -kq_- - 2k'q_- \end{aligned} \iff \begin{aligned} \ddot{q}_+ + \omega_+^2 q_+ &= 0 \\ \ddot{q}_- + \omega_-^2 q_- &= 0 \end{aligned} \qquad \begin{aligned} \omega_+^2 &= k/m \\ \omega_-^2 &= (k+2k')/m \geq \omega_+^2 \end{aligned}$$

Beats?

So now let's demand that R be a vector, not a matrix, and that the matrix multiplication become a vector multiplication.

$$\vec{Z} = x\hat{i} + y\hat{j}, \qquad R = \cos\phi\, I + \sin\phi\, \hat{k}\times \longrightarrow \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\begin{aligned} R\vec{Z} &= x\cos\phi\,\hat{i} + y\cos\phi\,\hat{j} + x\sin\phi\,\hat{k}\times\hat{i} + y\sin\phi\,\hat{k}\times\hat{j} \\ &= x\cos\phi\,\hat{i} + y\cos\phi\,\hat{j} + x\sin\phi\,\hat{j} + y\sin\phi\,(-\hat{i}) \\ &= x\cos\phi\,\hat{i}\hat{i} + y\cos\phi\,\hat{i}\hat{j} + x\sin\phi\,\hat{j}\hat{i} + y\sin\phi\,\hat{j}\hat{j} \end{aligned}$$

We need a multiplication table:

$$\hat{i}\hat{i} = \hat{i} \qquad \hat{i}\hat{j} = \hat{j} \qquad \text{cf.} \qquad \mathbb{1}\mathbb{1} = \mathbb{1} \qquad \mathbb{1}i = i$$

$$\hat{j}\hat{i} = \hat{k}\times\hat{i} = \hat{j} \qquad \hat{j}\hat{j} = \hat{k}\times\hat{j} = -\hat{i} \qquad\qquad i\mathbb{1} = i \qquad ii = -\mathbb{1}$$

$$z = x\,\mathbb{1} + y\,i = r(\cos\theta\,\mathbb{1} + \sin\theta\, i) = re^{i\theta} \implies z' = Rz = re^{i(\theta+\phi)}$$

$$R = \cos\phi\,\mathbb{1} + \sin\phi\, i = e^{i\phi}$$

Rotations: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} z' \\ z'^* \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix}$

2d real representation of the rotation group

two 1d representations of the rotation group

Once we allow complex numbers, all representations of commutative groups can be reduced to 1d representations.

SHO: $\begin{pmatrix} x \\ \dot{x}/\omega \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x_0 \\ \dot{x}_0/\omega \end{pmatrix} \iff \begin{pmatrix} z(t) \\ z^*(t) \end{pmatrix} = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix} \begin{pmatrix} z_0 \\ z_0^* \end{pmatrix}$

$$z = x + i\dot{x}/\omega$$

We often use z and say that $\mathrm{Re}(z)$ is the actual solution, but this is what we are actually doing.