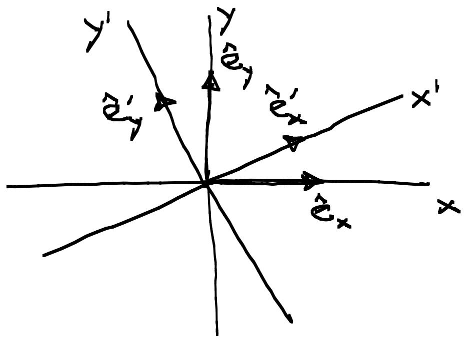
Phys 366

Lecture 5

Transformations of vectors

Transformations of vectors: Rotations and inversions

① Passive transformations



A <u>scalar</u> is a number that does not change under rotations.

A <u>vector</u> is a collection of 3 components that change under rotations according to

$$A_j' = \hat{e}_j' \cdot \vec{A} = \sum_k \underbrace{\hat{e}_j' \cdot \hat{e}_k}_{M_{jk}} A_k$$

$M_{jk} \leftarrow$ orthogonal matrix $MM^T = I$ (Why?)

Here the prime denotes the components of the fixed vector $\vec{A}$ in the new (primed) basis vectors.

Notice that

$$A_j = \hat{e}_j \cdot \vec{A} = \sum_k \hat{e}_j \cdot \hat{e}_k' A_k' = \sum_k M_{kj} A_k' = \sum_k M_{jk}^T A_k'$$

Matrix notation (representation):

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = M \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = M^T \begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix}$$

representation in the primed basis

representation in the original basis

The orthogonal matrices make up a group called $O(3)$.

$$1 = \det I = \det(MM^T) = (\det M)\underbrace{(\det M^T)}_{\det M} = (\det M)^2 \Rightarrow \det M = \pm 1$$

The orthogonal matrices with $\det M = +1$ are connected to the identity I and make up a group $SO(3)$. These are the rotations.

The orthogonal matrices with $\det M = -1$ are connected to $-I$ (parity or inversion). They are not a group. They are the rotations times inversion.

What about reflections?

(2) Active transformations

An active transformation is described by a (linear) operator R that maps one vector to another

$$\vec{A}' = R\vec{A}$$

Here the prime denotes the new vector that results from transforming $\vec{A}$

We can write this in terms of the way the components of $\vec{A}'$ are related to the components of $\vec{A}$:

$$A'_j = \hat{e}_j \cdot \vec{A}' = \hat{e}_j \cdot R\vec{A} = \sum_k \underbrace{\hat{e}_j \cdot R\hat{e}_k}_{\equiv R_{jk}} \underbrace{(\hat{e}_k \cdot \vec{A})}_{A_k} \qquad \begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix} = R \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Here the prime denotes the components of the new vector in the original basis

Summarize:

$$\vec{A}' = R\vec{A}, \quad R_{jk} = \hat{e}_j \cdot R\hat{e}_k, \quad A'_j = \sum_k R_{jk} A_k$$

Relation between passive and active transformations: Suppose we use R to generate the primed basis from the original basis, i.e.,

$$\underbrace{\hat{e}'_j \equiv R\hat{e}_j}_{\text{rotate } \hat{e}_j \text{ to get } \hat{e}'_j} = \sum_k \hat{e}_k \overbrace{\underbrace{(\hat{e}_k \cdot R\hat{e}_j)}_{\hat{e}'_j \cdot \hat{e}_k = M_{jk}}}^{R_{kj}} \implies M_{jk} = R_{kj} \quad \text{or} \quad M = R^T = R^{-1}$$

Why?

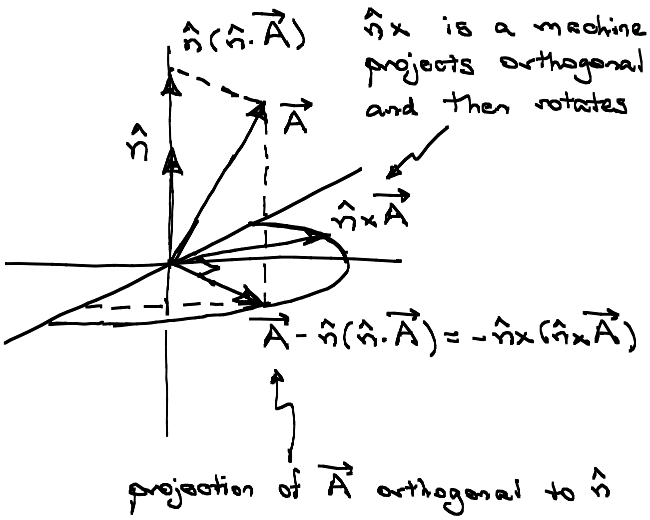
Matrix notation:

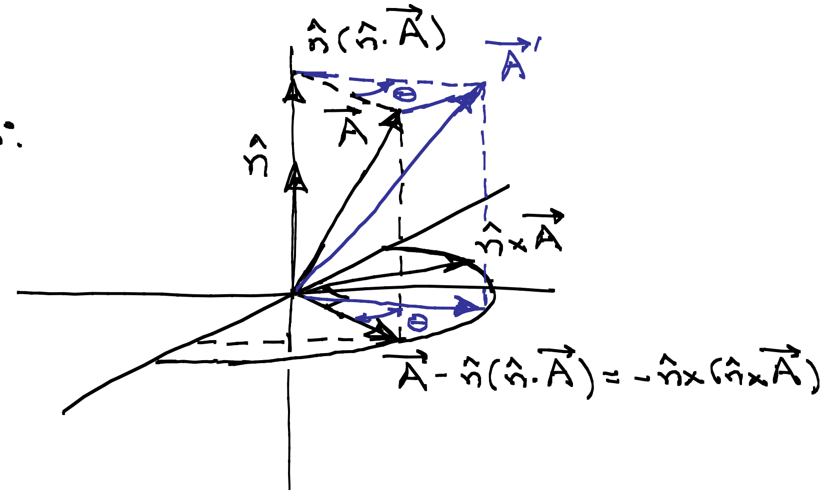
$$(\hat{e}'_x \ \ \hat{e}'_y \ \ \hat{e}'_z) = (\hat{e}_x \ \ \hat{e}_y \ \ \hat{e}_z) R \iff \begin{pmatrix} \hat{e}'_x \\ \hat{e}'_y \\ \hat{e}'_z \end{pmatrix} = M \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$

These two statements are transposes of one another, since transposition exchanges row and column vectors and $M = R^T$.

If we do a rotation R_1 followed by a rotation R_2, the overall rotation is $R = R_2 R_1$ or $M = R^T = R_1^T R_2^T = M_1 M_2$.

We can easily find the operator for a rotation by angle θ about an axis $\hat{n}$.





The operator that rotates vectors by angle θ about $\hat{n}$ is $R_{\hat{n}}(\theta)$:

$$\vec{A}' = R_{\hat{n}}(\theta)\vec{A} = \hat{n}(\hat{n}\cdot\vec{A}) + \cos\theta\left(\vec{A} - \hat{n}(\hat{n}\cdot\vec{A})\right) + \sin\theta\, \hat{n}\times\vec{A}$$
$$= \cos\theta\,\vec{A} + \hat{n}(\hat{n}\cdot\vec{A})(1-\cos\theta) + \sin\theta\,\hat{n}\times\vec{A}$$

rotation by θ about $\hat{n}$

$$R_{jk} = \hat{e}_j \cdot R_{\hat{n}}(\theta)\hat{e}_k = \delta_{jk}\cos\theta + n_j n_k(1-\cos\theta) - \epsilon_{jkl} n_l \sin\theta = M_{kj}$$

Vectors (polar vectors) and pseudovectors (axial vectors)

② Active

Parity P: inversion through origin

$$P\vec{A} = -\vec{A} \qquad \text{vector}$$

$$P\vec{C} = \vec{C} \qquad \text{pseudovector}$$

How can this be?

$$P(\underset{\uparrow}{\vec{A}} \times \underset{\uparrow}{\vec{B}}) = (-\vec{A})\times(-\vec{B}) = \vec{A}\times\vec{B}$$

polar vectors

① Passive: $M_{jk} = \hat{e}'_j \cdot \hat{e}_k$

Vector: $A'_j = M_{jk} A_k$

Pseudovector: $C'_j = (\det M)\, M_{jk} C_k$

pseudoscalar

$T' = (\det M)\, T$

changes sign under inversion

scalar triple product of polar vectors

Let's see how this works out for cross products:

polar vectors

$$\hat{e}'_j \cdot \vec{A}\times\vec{B} = \overbrace{\hat{e}'_j \cdot \hat{e}_k}^{M_{jk}}\, \epsilon_{klm} A_l B_m$$

$$= M_{jk}\, \epsilon_{klm} M_{nl} A'_n M_{pm} B'_p$$

$$= \underbrace{\epsilon_{klm} M_{jk} M_{nl} M_{pm}}_{\epsilon_{jnp}(\det M)} A'_n B'_p$$

$$= (\det M)\, \epsilon_{jkl} A'_k B'_l$$

This is a very tricky point. We will return to it when we study tensors.

$\vec{C} = \vec{A}\times\vec{B}$

Which of these is C'_j?

We deal with this by making the antisymmetric symbol a pseudo-tensor, called the Levi-Civita tensor, which transforms according to

$$\epsilon'_{jkl} = (\det M)\, M_{jm} M_{kn} M_{lp}\, \epsilon_{mnp} = \epsilon_{jkl}$$

With this choice, we have

$$C'_j = \epsilon'_{jkl} A'_k B'_l = \epsilon_{jkl} A'_k B'_l = (\det M)\, M_{jk} \underbrace{\epsilon_{klm} A_l B_m}_{C_k} = (\det M)\, M_{jk} C_k,$$

so $\vec{C}$ is a pseudovector.