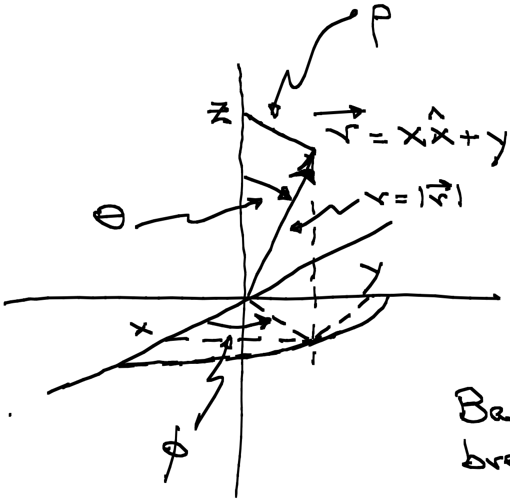
Phys 366
Lecture 6
Derivatives of vectors and curvilinear coördinates

Trajectories and curvilinear coördinates



position vector

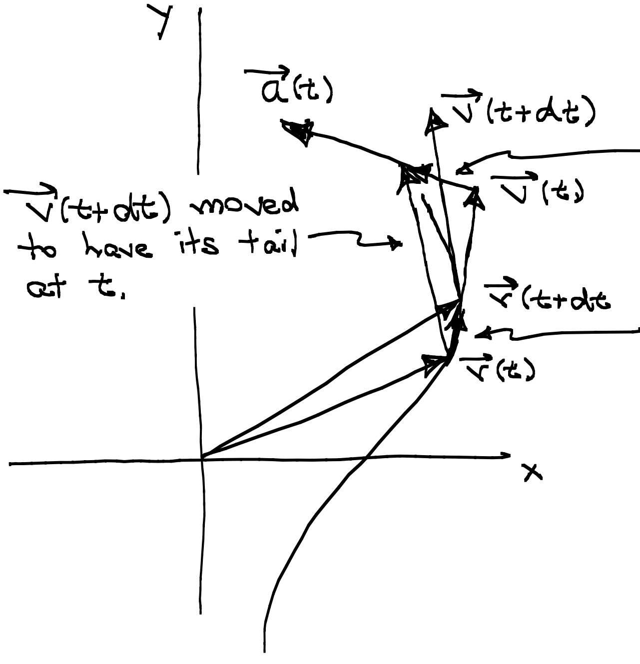
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = \rho\hat{\rho} + z\hat{z} = r\hat{r}$$

Basis vectors: all hell breaks loose.

Cartesian	Cylindrical	Spherical
$x = \rho\cos\phi = r\sin\theta\cos\phi$	ρ radial coördinate	r radial coördinate
$y = \rho\sin\phi = r\sin\theta\sin\phi$	ϕ azimuthal angle	θ polar angle
$z = z \quad = r\cos\theta$	z	ϕ azimuthal angle

use ρ when there is a chance of confusion with the spherical r; otherwise, use r

Trajectories, velocity, and acceleration



$d\vec{v} = \vec{v}(t+dt) - \vec{v}(t)$

$d\vec{r} = \vec{r}(t+dt) - \vec{r}(t)$

$\vec{r}(t) \longleftarrow$ trajectory: position as a function of

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}}$$

Cartesian coördinates: x, y, z

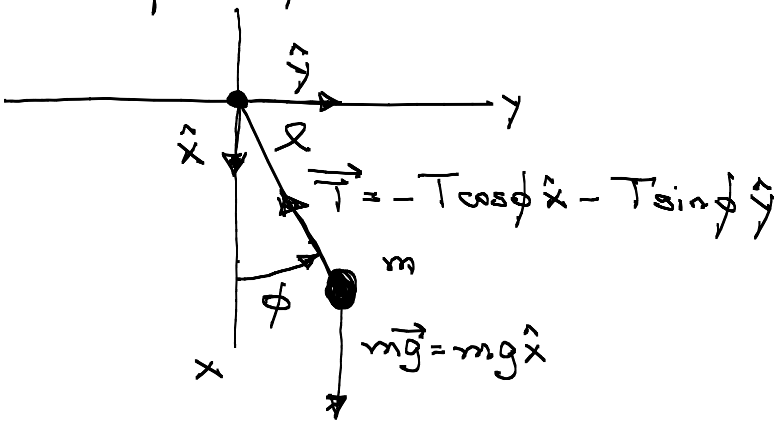
Trajectory: $x(t) = x_1(t)$, $y(t) = x_2(t)$, $z(t) = x_3(t)$

Position: $\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} = \sum_j x_j(t)\hat{e}_j$

Velocity: $\vec{v}(t) = \dot{\vec{r}} = \dot{x}(t)\hat{x} + \dot{y}(t)\hat{y} + \dot{z}(t)\hat{z} = \sum_j \dot{x}_j(t)\hat{e}_j$

Acceleration: $\vec{a}(t) = \dot{\vec{v}} = \ddot{x}(t)\hat{x} + \ddot{y}\hat{y} + \ddot{z}(t)\hat{z} = \sum_j \ddot{x}_j\hat{e}_j$

Example: pendulum



Newton's Law:

$m\vec{a} = m\vec{g} + \vec{T}$

Components:

$m\ddot{x} = mg - T\cos\phi$

$m\ddot{y} = -T\sin\phi$

$x = \ell\cos\phi$, $y = \ell\sin\phi$

$\dot{x} = -\ell\dot{\phi}\sin\phi$, $\dot{y} = \ell\dot{\phi}\cos\phi$

$\ddot{x} = -\ell\ddot{\phi}\sin\phi - \ell\dot{\phi}^2\cos\phi$, $\ddot{y} = \ell\ddot{\phi}\cos\phi - \ell\dot{\phi}^2\sin\phi$

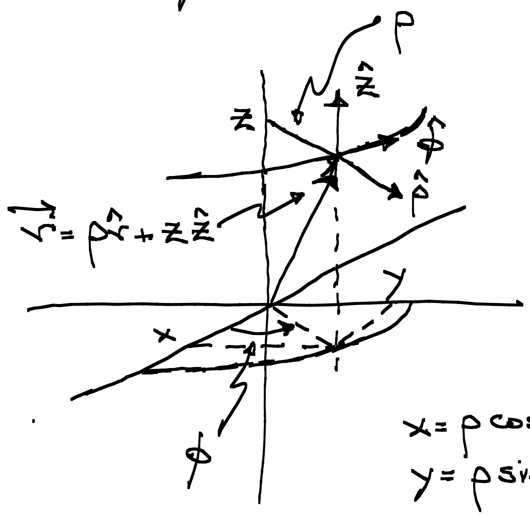
Equations of motion: ① $-\ell\ddot{\phi}\sin\phi - \ell\dot{\phi}^2\cos\phi = g - \frac{T}{m}\cos\phi$

② $\ell\ddot{\phi}\cos\phi - \ell\dot{\phi}^2\sin\phi = -\frac{T}{m}\sin\phi$

② $\times\cos\phi$ − ① $\times\sin\phi$: $\ddot{\phi} = -\frac{g}{\ell}\sin\phi$

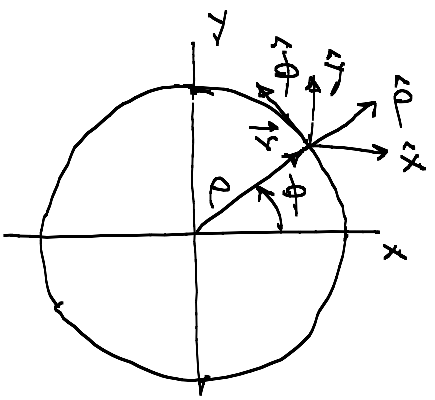
② $\times\sin\phi$ + ① $\times\cos\phi$: $T - mg\cos\phi = m\ell\dot{\phi}^2$

Cylindrical coördinates: ρ, ϕ, z



$x = \rho\cos\phi$

$y = \rho\sin\phi$

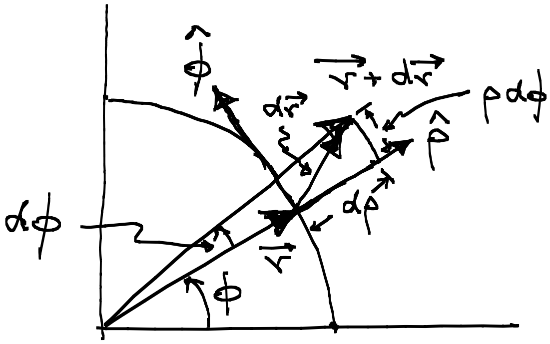


The cylindrical basis vectors depend on where you are. It is best to think of there being a different vector space at every point. To say what a vector's components are, you must say at what point it sits. The position vector is always regarded as sitting at the point it points to.

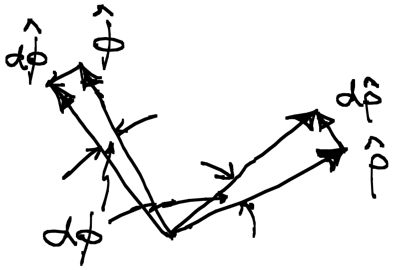
Besis-vector transformation

$$\hat\rho = \hat x\cos\phi + \hat y\sin\phi \qquad \hat x = \hat\rho\cos\phi - \hat\phi\sin\phi$$
$$\hat\phi = -\hat x\sin\phi + \hat y\cos\phi \qquad \hat y = \hat\rho\sin\phi + \hat\phi\cos\phi$$
$$\hat z = \hat z \qquad \hat z = \hat z$$

On to derivatives



$$\vec r = \rho\hat\rho + z\hat z$$
$$d\vec r = d\rho\,\hat\rho + \rho\underbrace{d\phi\,\hat\phi}_{d\hat\rho} + dz\,\hat z$$
$$\vec v = \frac{d\vec r}{dt} = \dot\rho\,\hat\rho + \rho\underbrace{\dot\phi\,\hat\phi}_{d\hat\rho/dt} + \dot z\,\hat z$$



$d\hat\rho$ and $d\hat\phi$ describe an infinitesimal rotation by $d\phi$:

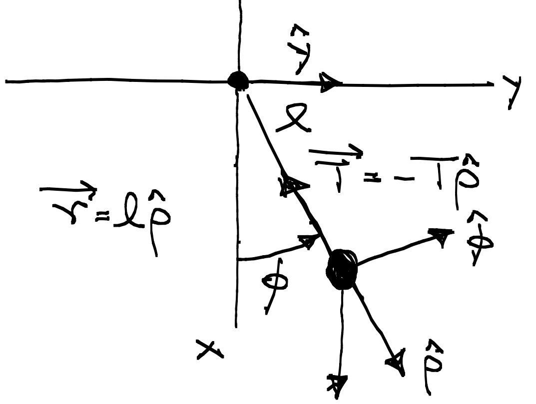
$$d\hat\rho = d\phi\,\hat\phi$$
$$d\hat\phi = -d\phi\,\hat\rho$$

$$\vec a = \dot{\vec v} = (\ddot\rho - \underset{\uparrow}{\rho\dot\phi^2})\hat\rho + (\rho\ddot\phi + \underset{\uparrow}{2\dot\rho\dot\phi})\hat\phi + \ddot z\,\hat z$$

centripetal acceleration

Coriolis acceleration

Example: pendulum



$$m\vec g = mg\cos\phi\,\hat\rho - mg\sin\phi\,\hat\phi$$

Newton's Law:

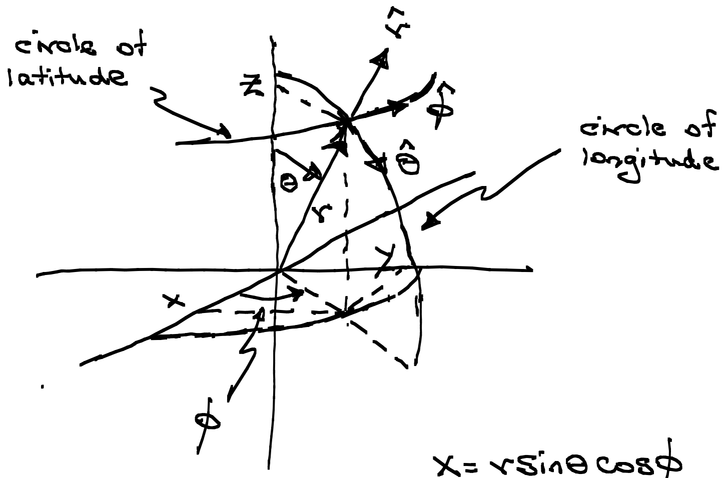
$$m\vec a = m\vec g + \vec T$$

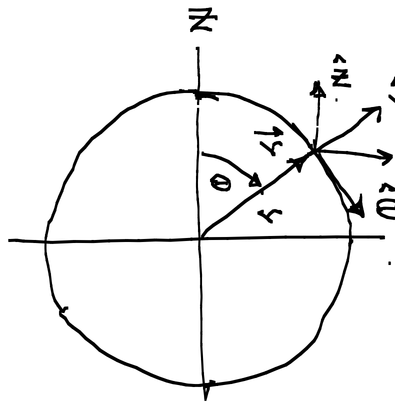
Components:

$$-m\ell\dot\phi^2 = mg\cos\phi - T$$
$$m\ell\ddot\phi = -mg\sin\phi$$

$$\Longrightarrow \quad T - mg\cos\phi = m\ell\dot\phi^2$$
$$\ddot\phi = -\frac{g}{\ell}\sin\phi$$

Spherical coördinates: r, θ, ϕ





$\hat{r} = \hat{z}\cos\theta + \hat{\rho}\sin\theta$

$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$

$\hat{\theta} = -\hat{z}\sin\theta + \hat{\rho}\cos\theta$

$$x = \underbrace{r\sin\theta}_{\rho}\cos\phi$$

$$y = \overbrace{r\sin\theta}\sin\phi$$

$$z = r\cos\theta$$

Basis-vector transformations:

$$\hat{r} = \hat{\rho}\sin\theta + \hat{z}\cos\theta = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

$$\hat{\theta} = \hat{\rho}\cos\theta - \hat{z}\sin\theta = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$

$$\hat{\phi} = \hat{\phi} \qquad = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\hat{\rho} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$$

$$\hat{\phi} = \hat{\phi}$$

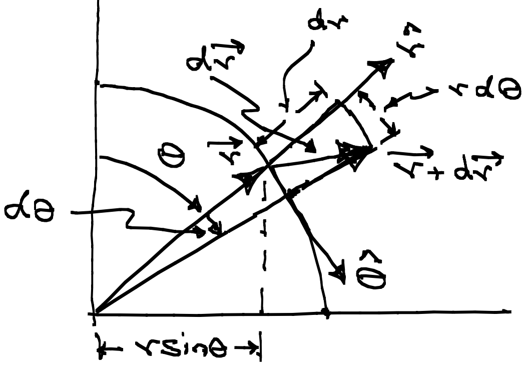
$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$

$$\hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{z} = \hat{z} \qquad = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

On to derivatives



$$\vec{r} = r\hat{r}$$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$d\hat{r} = d\theta\,\hat{\theta} + \sin\theta\,d\phi\,\hat{\phi}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} + \sin\theta\,\dot{\phi}\hat{\phi}$$

To get the acceleration, we need to know the derivatives of all the spherical basis vectors. To get these, you can think hard, draw pictures, differentiate $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ written in terms of $\hat{x}$, $\hat{y}$, $\hat{z}$, or use the kinetic energy written in spherical coördinates. Here's the answer:

$$d\hat{r} = d\theta\,\hat{\theta} + \sin\theta\, d\phi\,\hat{\phi}$$
$$d\hat{\theta} = -d\theta\,\hat{r} + \cos\theta\, d\phi\,\hat{\phi}$$
$$d\hat{\phi} = -\sin\theta\, d\phi\,\hat{r} - \cos\theta\, d\phi\,\hat{\theta}$$

This gives

$$\vec{a} = \dot{\vec{v}} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2)\,\hat{r}$$
$$+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2)\,\hat{\theta}$$
$$+ (r\sin\theta\,\ddot{\phi} + 2\sin\theta\,\dot{r}\dot{\phi} + 2r\cos\theta\,\dot{\theta}\dot{\phi})\,\hat{\phi}$$

I don't know anybody who has ever used this.

Torque and angular momentum

$$\vec{N} = \vec{r} \times \vec{F} = \hat{r} \times m\dot{\vec{v}} = \frac{d}{dt}(\underbrace{\vec{r} \times m\vec{v}}_{=\vec{L} = \vec{r} \times \vec{p}}) = \frac{d\vec{L}}{dt}$$

Newton's 2nd Law in torque-angular-momentum form

$$\vec{L} = m\vec{r} \times \vec{v} = mr\,\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi})$$
$$= mr^2(\dot{\theta}\,\hat{r}\times\hat{\theta} + \sin\theta\,\dot{\phi}\,\hat{r}\times\hat{\phi})$$
$$= \underbrace{mr^2}_{\substack{I \\ \text{moment of} \\ \text{inertia}}}\underbrace{(\dot{\theta}\hat{\phi} - \sin\theta\,\dot{\phi}\,\hat{\theta})}_{\substack{\vec{\omega} \\ \text{angular} \\ \text{velocity}}}$$

$$\vec{L} = I\vec{\omega}, \quad \vec{N} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(I\vec{\omega})$$

Pendulum: $r = l$, $\theta = \pi/2$, $I = ml^2$, $\vec{\omega} = -\dot{\phi}\,\hat{\theta} = \dot{\phi}\,\hat{z}$

$$\vec{N} = \vec{r} \times (m\vec{g} + \vec{T}) = \vec{r} \times m\vec{g} = mgl\,\underbrace{\hat{r}\times\hat{x}}_{-\hat{z}\sin\phi} = -mgl\sin\phi\,\hat{z}$$

$$\vec{N} = \frac{d}{dt}(I\vec{\omega}) \Longleftrightarrow -mgl\sin\phi = ml^2\ddot{\phi}$$
$$\ddot{\phi} = -\frac{g}{l}\sin\phi$$