Phys 366

Lecture 9

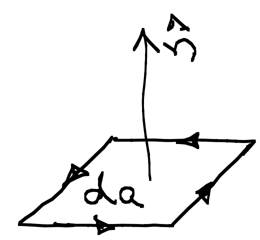
Vector fields and the curl

Vector field $\vec{F}$

$$\left(\begin{array}{c}\text{circulation of } \vec{F} \\ \text{around closed curve } C\end{array}\right) = \oint \vec{F} \cdot \vec{d\ell}$$

right-hand rule

$$\text{Curl } \vec{F} \cdot \hat{n}\, da = \nabla \times \vec{F} \cdot \hat{n}\, da = \oint_C \vec{F} \cdot \vec{d\ell}$$



Shape of area?

$\nabla \times \vec{F} \cdot \hat{n}$ is the differential version of circulation

Z- component: $da = dx\,dy$

$\hat{n} = \hat{x} \times \hat{y} = \hat{z}$

Components

Cartesian: $$\nabla \times \vec{F} \cdot \hat{z}\, da = \underbrace{F_y\left(x + \tfrac{1}{2}dx\right) dy - F_y\left(x - \tfrac{1}{2}dx\right) dy}_{\frac{\partial F_y}{dx}\, dx\, dy}$$

$$\underbrace{- F_x\left(y + \tfrac{1}{2}dy\right) dx + F_x\left(y - \tfrac{1}{2}dy\right) dx}_{-\frac{\partial F_x}{\partial y}\, dy\, dx}$$

$$= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) da$$

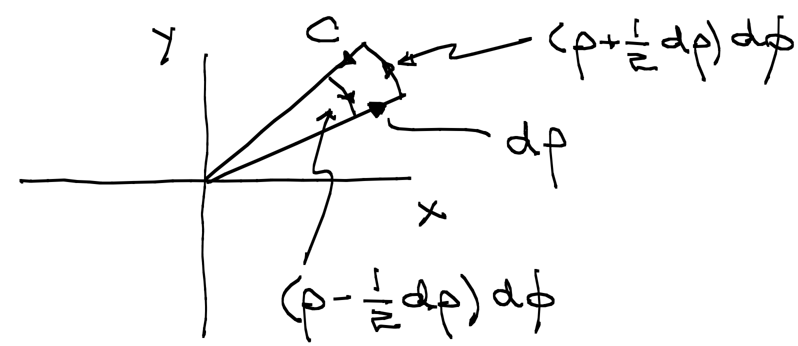
$$\nabla \times \vec{F} \cdot \hat{z} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

So we have (∇ operator)

$$\nabla \times \vec{F} = \hat{e}_j \epsilon_{jkl} F_{l,k} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

$F_{l,k} \to \partial F_l / \partial x_k$

$\nabla \times \vec{F}$ is a pseudovector because ∇ transforms like a vector.



Cylindrical:

$$\nabla \times \vec{F} \cdot \hat{z}\, \rho\, d\rho\, d\phi = \oint_C \vec{F} \cdot \vec{dl}$$

$$= \underbrace{F_\rho(\phi - \tfrac{1}{2} d\phi)\, d\rho - F_\rho(\phi + \tfrac{1}{2} d\phi)\, d\rho}_{-\frac{\partial F_\rho}{\partial \phi} d\phi\, d\rho \;=\; -\frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi} \rho\, d\rho\, d\phi}$$

$$+ \underbrace{F_\phi(\rho + \tfrac{1}{2} d\rho)(\rho + \tfrac{1}{2} d\rho)\, d\phi - F_\phi(\rho - \tfrac{1}{2} d\rho)(\rho - \tfrac{1}{2} d\rho)\, d\phi}_{\frac{\partial(\rho F_\phi)}{\partial \rho} d\rho\, d\phi \;=\; \frac{1}{\rho}\frac{\partial(\rho F_\phi)}{\partial \rho} \rho\, d\rho\, d\phi}$$

This already shows the two effects. Given a component of $\nabla\times\vec{F}$ and a coördinate for a derivative, we have to worry about (i) change in the line element for the third coördinate from the front to the back of C and (ii) converting the derivative coördinate to length. So we get

Cylindrical: $$\nabla\times\vec{F} = \hat{\rho}\left(\underset{\substack{dz\\ \rho\,d\phi}}{\frac{1}{\rho}\frac{\partial F_z}{\partial\phi}} - \underset{\substack{\rho\,d\phi\\ dz}}{\frac{\partial F_\phi}{\partial z}}\right) + \hat{\phi}\left(\underset{\substack{d\rho\\ dz}}{\frac{\partial F_\rho}{\partial z}} - \underset{\substack{dz\\ d\rho}}{\frac{\partial F_z}{\partial\rho}}\right) + \hat{z}\left(\underset{\substack{\rho\,d\phi\\ d\rho}}{\frac{1}{\rho}\frac{\partial(\rho F_\phi)}{\partial\rho}} - \underset{\substack{d\rho\\ \rho\,d\phi}}{\frac{1}{\rho}\frac{\partial F_\rho}{\partial\phi}}\right)$$

General: $$\nabla\times\vec{F} = \hat{x}_1\left(\underset{\substack{h_3\,dx_3\\ h_2\,dx_2}}{\frac{1}{h_2h_3}\frac{\partial(h_3F_3)}{\partial x_2}} - \underset{\substack{h_2\,dx_2\\ h_3\,dx_3}}{\frac{1}{h_2h_3}\frac{\partial(h_2F_2)}{\partial x_3}}\right)$$

length for 3rd coördinate

length for derivative coördinate

$$+\hat{x}_2\left(\underset{\substack{h_1\,dx_1\\ h_3\,dx_3}}{\frac{1}{h_1h_3}\frac{\partial(h_1F_1)}{\partial x_3}} - \underset{\substack{h_3\,dx_3\\ h_1\,dx_1}}{\frac{1}{h_1h_3}\frac{\partial(h_3F_3)}{\partial x_1}}\right)$$

$$+\hat{x}_3\left(\underset{\substack{h_2\,dx_2\\ h_1\,dx_1}}{\frac{1}{h_1h_2}\frac{\partial(h F_2)}{\partial x_1}} - \underset{\substack{h_1\,dx_1\\ h_2\,dx_2}}{\frac{1}{h_1h_2}\frac{\partial(h_1F_1)}{\partial x_2}}\right)$$

<u>OR</u> $$\nabla \times \vec{F} = \sum_{j,k,\ell} \hat{x}_j \, \epsilon_{jk\ell} \frac{1}{h_k h_\ell} \left(\frac{\partial (h_\ell F_\ell)}{\partial x_k} - \frac{\partial (h_k F_k)}{\partial x_\ell} \right)$$

Spherical: $$\nabla \times \vec{F} = \hat{r} \left(\frac{1}{r\sin\theta} \frac{\partial (\sin\theta F_\phi)}{\partial \theta} - \frac{1}{r\sin\theta} \frac{\partial F_\theta}{\partial \phi} \right) + \hat{\theta} \left(\frac{1}{r\sin\theta} \frac{\partial F_r}{\partial \phi} - \frac{1}{r} \frac{\partial (r F_\phi)}{\partial r} \right)$$

$\frac{r\sin\theta\, d\phi}{r\,d\theta}$ $\quad \frac{r\,d\theta}{r\sin\theta\, d\phi}$ $\quad \frac{dr}{r\sin\theta\, d\phi}$ $\quad \frac{r\sin\theta\, d\phi}{dr}$

$$+ \hat{\phi} \left(\frac{1}{r} \frac{\partial (r F_\theta)}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \theta} \right)$$

$\frac{r\,d\theta}{dr}$ $\quad \frac{dr}{r\,d\theta}$

Stokes's theorem: $$\int_S \nabla \times \vec{F} \cdot \vec{da} = \oint_C \vec{F} \cdot \vec{d\ell}$$

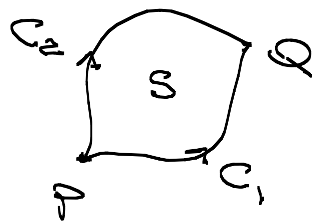
right-hand rule

Boundary of a boundary is zero:

① $$\nabla \times \nabla f = \hat{e}_j \epsilon_{jk\ell} (\nabla f)_{\ell,k} = \hat{e}_j \epsilon_{jk\ell} f_{,\ell k} = 0$$

$$\Longleftrightarrow \int_S \nabla \times \nabla f \cdot \vec{da} = \oint_C \nabla f \cdot \vec{d\ell} = 0$$

a closed curve has no endpoints



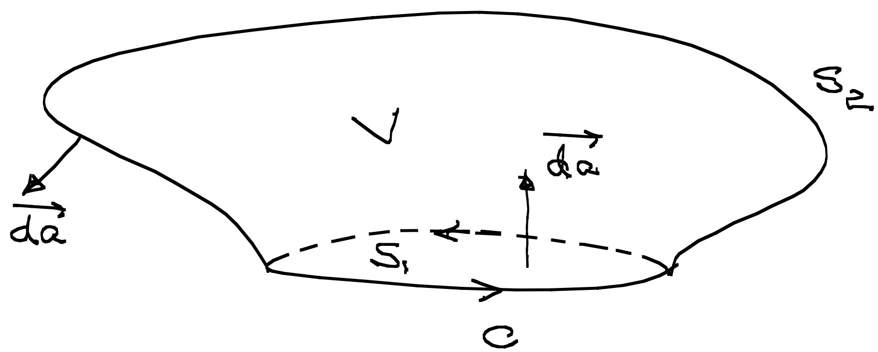
$$\underbrace{\int_{C_2} \nabla f \cdot \vec{dl} - \oint_{C_1} \nabla f \cdot \vec{dl}}_{= f(Q) - f(P)} = \oint_S \nabla \times \nabla f \cdot \vec{da} = 0$$

(2)

$$\nabla \cdot \nabla \times \vec{F} = (\nabla \times \vec{F})_{j,j} = \epsilon_{jkl} F_{l,kj} = 0$$

$$\Longleftrightarrow \int_V \nabla \cdot \nabla \times \vec{F} = \oint_S \nabla \times \vec{F} \cdot \vec{da} = 0$$

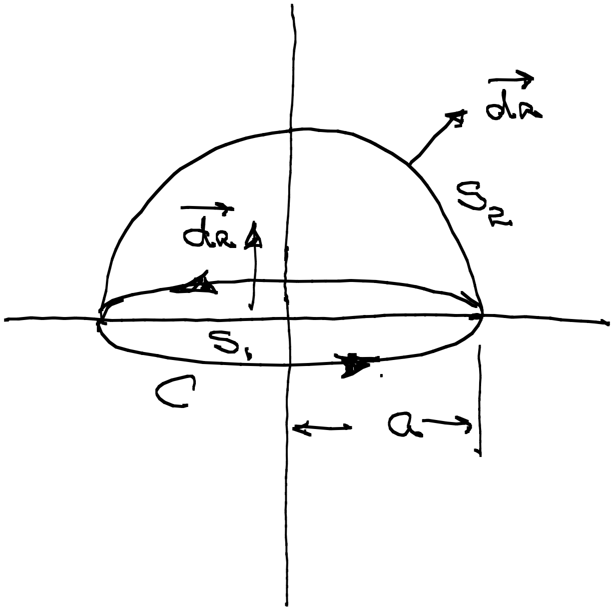
a closed surface has no bounding curve



$$\underbrace{\int_{S_2} \nabla \times \vec{F} \cdot \vec{da} - \int_{S_1} \nabla \times \vec{F} \cdot \vec{da}}_{= \int_C \vec{F} \cdot \vec{dl}} = \int_V \nabla \cdot \nabla \times \vec{F} \, dV = 0$$

Example: $\vec{F} = F(r)\hat{\phi}$, $\nabla\times\vec{F} = \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F(r)) - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial r}(rF(r))$

$$= \hat{r}\,\frac{\cos\theta}{\sin\theta}\frac{F(r)}{r} - \hat{\theta}\frac{1}{r}\frac{d}{dr}(rF(r))$$



$$\oint_C \vec{F}\cdot\underset{\uparrow\atop a\,d\phi\,\hat{\phi}}{\vec{dl}} = \int_0^{2\pi} d\phi\; aF(a) = 2\pi a F(a)$$

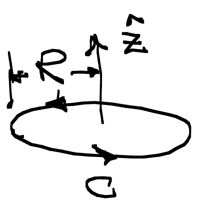
$$\int_{S_1} \nabla\times\vec{F}\cdot\underset{\uparrow\atop -\hat{\theta}\,dr\,r\,d\phi}{\vec{da}} = \underbrace{\int_0^a dr\frac{d}{dr}(rF(r))}_{aF(a)}\underbrace{\int_0^{2\pi} d\phi}_{2\pi} = 2\pi a F(a)$$

$$\int_{S_2} \nabla\times\vec{F}\cdot\underset{\uparrow\atop \hat{r}\,a^2\sin\theta\,d\theta\,d\phi}{\vec{da}} = aF(a)\underbrace{\int_0^{\pi/2} d\theta\cos\theta}_{\sin\theta\big|_0^{\pi/2} = 1}\underbrace{\int_0^{2\pi} d\phi}_{2\pi} = 2\pi a F(a)$$

Fluid flow: Let $\vec{\omega} = \omega\hat{z}$, $\vec{v} = \vec{\omega}\times\vec{r} = \omega r\sin\theta\,\hat{\phi} = \omega\rho\,\hat{\phi}$, $\nabla\times\vec{v} = 2\omega\rho\,\hat{z}$

$$\int_S \nabla\times\vec{v}\cdot\hat{z}\,da = \oint_C \vec{v}\cdot\underset{\uparrow\atop R\,d\phi\,\hat{\phi}}{\vec{dl}} = \int_0^{2\pi} v_\phi\,R\,d\phi = 2\pi\omega R^2$$

$$= 2\omega\int_0^R d\rho\;\rho^2\int_0^{2\pi} d\phi = 2\pi\omega R^2$$



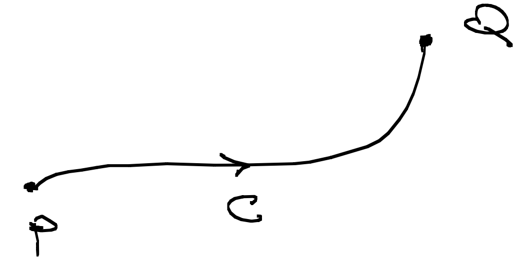
$da = \rho\,d\rho\,d\phi$

$$\therefore\quad \frac{\int_S \nabla\times\vec{v}\cdot\vec{da}}{\pi R^2} = 2\omega$$

Vortex: the vortex line extends to infinity in both directions, even when it's not a straight line.

This kind of flow, in which the fluid rotates about a line, is called a *vortex*. Generally, $\nabla \times \vec{v}$ is called the vorticity of the flow. Flow without rotation is conservative, i.e., $\nabla \times \vec{v} = 0$.

A *conservative* field is one that satisfies $\nabla \times \vec{F} = 0$. This implies that $\vec{F} = \nabla f$ for some scalar field f. You construct f as follows



$$f(Q) = f(P) + \int_C \vec{F} \cdot \vec{dl}$$

choose arbitrarily

independent of path C by Stokes's theorem

$$\nabla \cdot \vec{F} = \nabla \cdot \nabla f = \nabla^2 f$$

$\nabla^2 = \nabla \cdot \nabla$ is called the *Laplacian*. It arises frequently, since it is the natural second derivative of a scalar field.

Vector differential identities:

$$\nabla(fg) = g\nabla f + f\nabla g \qquad\qquad \nabla(\vec{F}\cdot\vec{G})\ ?$$

$$\nabla\cdot(f\vec{F}) = \nabla f\cdot\vec{F} + f\nabla\cdot\vec{F}$$

$$\nabla\cdot(\vec{F}\times\vec{G}) = \frac{\partial}{\partial x^j}(\epsilon_{jkl}F_k G_l)$$

$$= \epsilon_{jkl}F_{k,j}G_l + \epsilon_{jkl}F_k G_{l,j}$$

$$= G_l\,\epsilon_{ljk}F_{k,j} - F_k\,\epsilon_{kjl}G_{l,j}$$

$$= \vec{G}\cdot\nabla\times\vec{F} - \vec{F}\cdot\nabla\times\vec{G}$$

$$\nabla\times(f\vec{F}) = \nabla f\times\vec{F} + f\nabla\times\vec{F}$$

Now for two hard ones:

1. $$\nabla\times(\vec{F}\times\vec{G}) = \hat{e}_j\,\epsilon_{jkl}\underbrace{(\vec{F}\times\vec{G})_{l,k}}_{(\epsilon_{lmn}F_m G_n)_{,k} = \epsilon_{lmn}(F_{m,k}G_n + F_m G_{n,k})}$$

$$= \hat{e}_j\,\underbrace{\epsilon_{jkl}\,\epsilon_{lmn}}_{\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}}(F_{m,k}G_n + F_m G_{n,k})$$

$$= \hat{e}_j \left(F_{j,k} G_k - F_{k,k} G_j + F_j G_{k,k} - F_k G_{j,k} \right)$$

$$\nabla \times (\vec{F} \times \vec{G}) = \underbrace{(\vec{G} \cdot \nabla)\vec{F}}_{} - (\vec{F} \cdot \nabla)\vec{G} + \vec{F}\, \nabla \cdot \vec{G} - \vec{G}\, \nabla \cdot \vec{F}$$

this set of symbols means $G_k \nabla_k (F_j \hat{e}_j) = \hat{e}_j G_k F_{j,k}$

(e) $$\nabla (\vec{F} \cdot \vec{G}) = \hat{e}_j (F_k G_k)_{,j}$$

$$= \underbrace{\hat{e}_j F_{k,j} G_k}_{} + \hat{e}_j F_k G_{k,j}$$

$$= \hat{e}_j \left(F_{k,j} - F_{j,k} + F_{j,k} \right) G_k$$

$$= \hat{e}_j (F_{k,j} - F_{j,k}) G_k + \hat{e}_j G_k F_{j,k}$$

Use

$\epsilon_{\ell jk} \epsilon_{\ell mn} F_{n,m} = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) F_{n,m}$ $\longrightarrow$ $= \hat{e}_j\, \epsilon_{\ell jk} \epsilon_{\ell mn} F_{n,m} G_k + \hat{e}_j G_k F_{j,k}$

$= F_{k,j} - F_{j,k}$

$$= \hat{e}_j\, \epsilon_{jk\ell} G_k \epsilon_{\ell mn} F_{n,m} + \hat{e}_j G_k F_{j,k}$$

$$= \vec{G} \times (\nabla \times \vec{F}) + (\vec{G} \cdot \nabla)\vec{F}$$

So $$\nabla (\vec{F} \cdot \vec{G}) = \vec{G} \times (\nabla \times \vec{F}) + \vec{F} \times (\nabla \times \vec{G}) + (\vec{G} \cdot \nabla)\vec{F} + (\vec{F} \cdot \nabla)\vec{G}$$