Phys 366

Lectures 14 - 15

Tensors

Who ordered tensors?

I. Inertia tensor.

Dynamics. Collection of particles with mass m_j, position $\vec{r}_j$, velocity $\vec{v}_j = \dot{\vec{r}}_j$, momentum $\vec{p}_j = m_j \vec{v}_j$. $\vec{F}_j$ is the <u>external</u> force on particle j, and $\vec{f}_{jk}$ is the <u>internal</u> force on particle j due to particle k. Newton's 3rd Law says that $\vec{f}_{jk} = -\vec{f}_{kj}$

Total momentum: $\vec{P} = \sum_j \vec{p}_j$

① $$\frac{d\vec{P}}{dt} = \sum_j \dot{\vec{p}}_j \underset{\uparrow \text{ Newton's 2nd Law}}{=} \sum_j \vec{F}_j + \underbrace{\sum_{j,k} \vec{f}_{jk}}_{=0 \text{ by 3rd Law}} = \vec{F} = \begin{pmatrix}\text{total external}\\ \text{force}\end{pmatrix}$$

Total angular momentum: $\vec{L} = \sum_j \vec{r}_j \times \vec{p}_j$

$$\frac{d\vec{L}}{dt} = \sum_j \underbrace{\dot{\vec{r}}_j \times \vec{p}_j}_{0} + \vec{r}_j \times \dot{\vec{p}}_j \underset{\uparrow \text{ Newton's 2nd Law}}{=} \sum_j \vec{r}_j \times \vec{F}_j + \underbrace{\sum_{j,k} \vec{r}_j \times \vec{f}_{jk}}$$

$$= \sum_{\substack{j<k \\ \text{pairs}}} \vec{r}_j \times \vec{f}_{jk} + \vec{r}_k \times \vec{f}_{kj}$$

$$\text{3rd Law} \rightarrow = \sum_{j<k} (\vec{r}_j - \vec{r}_k) \times \vec{f}_{jk}$$

$= 0$ for 3rd Law central forces

② $$\frac{d\vec{L}}{dt} = \sum_j \vec{r}_j \times \vec{F}_j = \vec{N} = \begin{pmatrix}\text{total external}\\ \text{torque}\end{pmatrix}$$

Rigid bodies: particles maintain their relative orientation
common displacement plus common rotation
① and ② are sufficient.

What you're used to: torque and angular momentum about one axis.

Dynamics

$$N_z = (\ell\hat{\rho}\times F\hat{\phi})_z = \ell F = \dot{L}_z$$

Kinematics

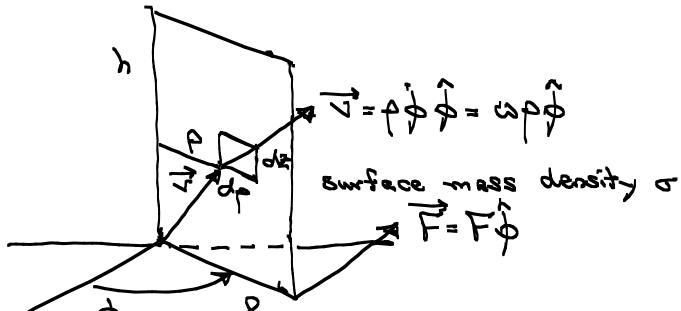
$$L_z = \int \sigma\, d\rho\, dz\, \underbrace{(\vec{r}\times\vec{v})_z}_{(\rho\hat{\rho}\times\omega\rho\hat{\phi})_z = \omega\rho^2}$$

$$= \omega\sigma \underbrace{\int_0^\ell \rho^2 d\rho}_{\frac{1}{3}\ell^3} \underbrace{\int_0^h dz}_{h}$$

$$= \frac{1}{3}\underbrace{\sigma\ell h}_{M}\,\ell^2\,\omega$$

$$L_z = \frac{1}{3}M\ell^2\omega = I\omega$$

moment of inertia



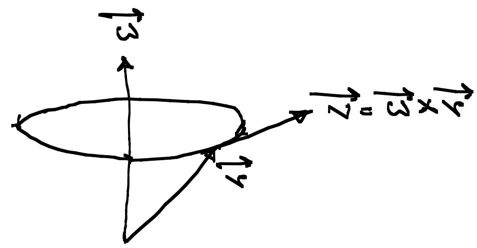
Door swinging on a hinge

Equation of motion

$$F\ell = N_z = \dot{L}_z = I\dot{\omega} = I\ddot{\phi}$$

But the kinematics is generally more complicated than this. For an arbitrary rigid body, we have

$$\vec{p} = \int dm\,\vec{v} = \int dm\,\vec{\omega}\times\vec{r}$$



$$\vec{L} = \int dm\,\vec{r}\times\vec{v}$$

$$= \int \rho\, dV\, \vec{r}\times(\vec{\omega}\times\vec{r}) \longrightarrow \vec{\omega}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{r}\cdot\vec{\omega})$$

mass density

$$= r^2\vec{\omega} - \vec{r}(\vec{r}\cdot\vec{\omega})$$

$$L_j = \int dV\,\rho\,(r^2\omega_j - x_j x_k \omega_k) = \omega_k \underbrace{\int dV\,\rho\,(r^2\delta_{jk} - x_j x_k)}_{I_{jk}} = I_{jk}\omega_k = L_j$$

$$I_{jj} = \int dV\,\rho\,(r^2 - x_j^2)$$

$I_{jk} \longleftarrow$ components of inertia tensor

$$I_{jk} = -\int dV\,\rho\, x_j x_k$$

The inertia tensor is a symmetric, rank-2 tensor. It is a linear operator on vectors to vectors.

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$I_{xx} = \int dV \rho (y^2 + z^2)$$
$$I_{yy} = \int dV \rho (x^2 + z^2)$$
$$I_{zz} = \int dV \rho (x^2 + y^2)$$
$$I_{xy} = -\int dV \rho xy = I_{yx}$$
$$I_{xz} = -\int dV \rho xz = I_{zx}$$
$$I_{yz} = -\int dV \rho yz = I_{zy}$$

Working abstractly, we have

unit tensor
Kronecker delta

$$\vec{L} = \underbrace{\int dV \rho (r^2 \overleftrightarrow{\delta} - \vec{r} \otimes \vec{r})}_{\overleftrightarrow{I}} \cdot \vec{\omega}$$

$$\overleftrightarrow{I} = \int dV \rho (r^2 \overleftrightarrow{\delta} - \vec{r} \otimes \vec{r})$$

$$[\overleftrightarrow{I}] = ML^2$$

tensor OR direct product OR outer product

$$= \sum_{j,k} \hat{e}_j \otimes \hat{e}_k \underbrace{(\hat{e}_j \cdot \overleftrightarrow{I} \cdot \hat{e}_k)}_{I_{jk}}$$

Bra-ket version

unit operator

$$I = \int \rho \, dV \left(\langle \vec{r} | \vec{r} \rangle \delta - | \vec{r} \rangle \langle \vec{r} | \right)$$

$$= \sum_{j,k} |e_j\rangle \underbrace{\langle e_j | I | e_k \rangle}_{I_{jk}} \langle e_k |$$

$\overleftrightarrow{I}$ can also be thought of as a bilinear map from two vectors to a scalar: $\vec{v} \cdot \overleftrightarrow{I} \cdot \vec{w} = \langle \vec{v} | I | \vec{w} \rangle$. Indeed, the kinetic energy of a rigid body is

$$K = \frac{1}{2} \int \rho \, dV \, \underbrace{\vec{v} \cdot \vec{v}}_{= (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) = \vec{\omega} \cdot [\vec{r} \times (\vec{\omega} \times \vec{r})]} = \frac{1}{2} \vec{\omega} \cdot \overbrace{\int \rho \, dV \, \vec{r} \times (\vec{\omega} \times \vec{r})}^{\vec{L}} = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$$

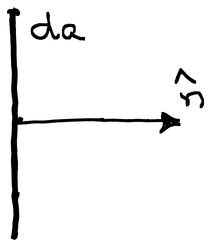
$$K = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} = \frac{1}{2} \langle \vec{\omega} | I | \vec{\omega} \rangle = (\omega_1 \;\; \omega_2 \;\; \omega_3) \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

We'll work on these later.

- Transformations: change of origin (parallel-axis theorem) and orthogonal transformations
- Diagonalization, principal axes, and principal moments of inertia

II. Stress tensor

The stress tensor $\overleftrightarrow{T}$ quantifies the flow of momentum across a surface. Since momentum flow is a force, $\overleftrightarrow{T}$ quantifies the <u>force</u> across surfaces.



$$\begin{pmatrix}\text{momentum that flows}\\ \text{across } \overrightarrow{da} \text{ in time } dt\end{pmatrix} = \overleftrightarrow{T}\cdot\overrightarrow{da}\,dt = dt\sum_j \hat{e}_j\left(\hat{e}_j\cdot\overleftrightarrow{T}\cdot\hat{n}\right)da$$

$$= dt\sum_j \hat{e}_j\,T_{jk}\,n_k\,da$$

symmetric 2-tensor

$$[\overleftrightarrow{T}] = \frac{\text{force}}{\text{area}} = \frac{M}{T^2 L}$$

symmetry guaranteed by angular-momentum conservation

$$\begin{pmatrix}\text{force left side exerts}\\ \text{on right side}\end{pmatrix} = \begin{pmatrix}\text{rate of momentum}\\ \text{flow across } \overrightarrow{da}\end{pmatrix} = \overleftrightarrow{T}\cdot\overrightarrow{da}$$

automatically 3rd Law

Force on a volume:

$$\vec{F} = -\oint_S \underbrace{\overleftrightarrow{T}\cdot\overrightarrow{da}}_{\hat{e}_j T_{jk}\hat{e}_k\cdot\overrightarrow{da}} = -\int_V \underbrace{\nabla\cdot\overleftrightarrow{T}}_{\hat{e}_j T_{jk,k}}\,d\tau$$

$-\nabla\cdot\overleftrightarrow{T}$ is the <u>force density</u>

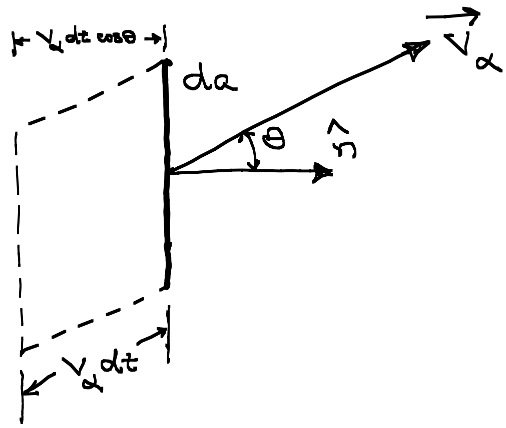
Extended example from transport theory.

What about particles with different masses, like the various molecules in air?

Particles carrying some property m, e.g., mass. At a particular point $\vec{r}$, divide the particles up into types. Particles of type α have number density n_α and have velocity $\vec{v}_\alpha$. The overall number density is $n = \sum_\alpha n_\alpha$, and the mass density is $\rho = \sum_\alpha m n_\alpha = mn$. The average velocity is

$$\vec{v}(\vec{r}) = \langle \vec{v} \rangle = \frac{1}{n} \sum_\alpha n_\alpha \vec{v}_\alpha \longleftrightarrow \frac{1}{n} \int d^3v \, n(\vec{r}, \vec{v}) \, \vec{v}$$

Particles of type α:



$$\begin{pmatrix}\text{number of particles} \\ \text{of type } \alpha \text{ that pass} \\ \text{through } da \text{ in } dt\end{pmatrix} = n_\alpha \, da \, v_\alpha \, dt \cos\theta$$
$$= n_\alpha \vec{v}_\alpha \cdot \vec{da} \, dt$$

$$\begin{pmatrix}\text{mass current} \\ \text{through } \vec{da}\end{pmatrix} = \sum_\alpha m n_\alpha \vec{v}_\alpha \cdot \vec{da}$$
$$= \rho \vec{v} \cdot \vec{da}$$

$\rho\vec{v}$: mass current density

$$\begin{pmatrix}\text{mass current} \\ \text{through } S\end{pmatrix} = \int_S \rho \vec{v} \cdot \vec{da}$$

Sign issues: sign of property
direction of flow

Conservation of mass:

① Integral form

$$\underbrace{\frac{d}{dt} \int_V \rho \, d\tau}_{\int_V \frac{\partial \rho}{\partial t} d\tau} = - \underbrace{\oint_S \rho \vec{v} \cdot \vec{da}}_{\int_V \nabla \cdot (\rho \vec{v}) d\tau} + \int_V R \, d\tau$$

R: source ($R > 0$) or sink ($R < 0$) rate

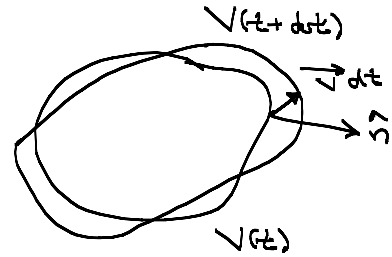
Differential form: $\dfrac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v}) + R$

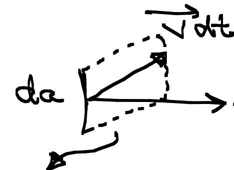
$R = 0$ is the conservation of mass

(2) Integral form:

$$\frac{d}{dt}\int_{V(t)} \rho \, d\tau = \int_{V(t)} R \, d\tau$$

$V(t)$ ← comoving volume (follows the flow)





$$(\text{added volume}) = da\,\vec{v}\cdot\hat{n}\,dt = \vec{v}\cdot\vec{da}\,dt$$

$$\frac{d}{dt}\int_{V(t)} d\tau = \frac{dV}{dt} = \oint_S \vec{v}\cdot\vec{da} = \int_V \nabla\cdot\vec{v}\,d\tau$$

Differential form:

$$R\Delta V = \underset{\substack{\uparrow \\ \text{differential volume that follows the flow}}}{\frac{d}{dt}(\rho\Delta V)} = \frac{d\rho}{dt}\Delta V + \rho \underbrace{\frac{d\Delta V}{dt}}_{=\,\Delta V\,\nabla\cdot\vec{v}} = \Delta V\left(\frac{d\rho}{dt} + \rho\nabla\cdot\vec{v}\right)$$

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\vec{v} + R$$

Why is this the same?

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \sum_j \frac{\partial\rho}{\partial x_j}\underbrace{\frac{dx_j}{dt}}_{v_j} = \frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho$$

$$\text{So}\quad \frac{\partial\rho}{\partial t} = \underbrace{-\vec{v}\cdot\nabla\rho - \rho\nabla\cdot\vec{v}}_{-\nabla\cdot(\rho\vec{v})} + R$$

But what if the "property" carried by the particles is a vector and depends on the velocity, e.g., the momentum. Then the current is a vector, and the current density has to have two indices, one to dot into $\vec{da}$ and the other to be the component of the momentum current. Thus the momentum current density is a rank-two tensor.

$$\begin{pmatrix}\text{momentum that passes} \\ \text{through } \vec{da} \text{ in } dt\end{pmatrix} = dt\sum_\alpha \vec{p}_\alpha n_\alpha \vec{v}_\alpha\cdot\vec{da}$$

$$= dt\sum_\alpha m n_\alpha \vec{v}_\alpha\,(\vec{v}_\alpha\cdot\vec{da})$$

$$\underbrace{\frac{1}{dt}\left(\begin{array}{l}\text{momentum that passes}\\ \text{through } \vec{da} \text{ in } dt\end{array}\right)}_{\substack{\text{force that } -\hat{n} \text{ side exerts}\\ \text{on } +\hat{n} \text{ side of } \vec{da}}} = \underbrace{\left(\sum_\alpha m n_\alpha \vec{v}_\alpha \otimes \vec{v}_\alpha\right)}_{\substack{\text{momentum current density OR}\\ \text{force per area OR}\\ \text{stress tensor}}} \cdot \vec{da}$$

This force automatically satisfies Newton's 3rd Law

There is an important difference between the momentum current density and the mass current density, having to do with the average velocity $\vec{v} = \frac{1}{n}\sum_\alpha n_\alpha \vec{v}_\alpha$. Let's write

$$\vec{v}_\alpha = \vec{v} + \underbrace{\delta\vec{v}_\alpha}_{\substack{\text{deviation from}\\ \text{average flow}}} \qquad \sum_\alpha n_\alpha \delta\vec{v}_\alpha = 0$$

$$\sum_\alpha m n_\alpha \vec{v}_\alpha \otimes \vec{v}_\alpha = \sum_\alpha m n_\alpha (\vec{v} + \delta\vec{v}_\alpha) \otimes (\vec{v} + \delta\vec{v}_\alpha)$$

$$= \rho\vec{v}\otimes\vec{v} + m\sum_\alpha n_\alpha \delta\vec{v}_\alpha \otimes \delta\vec{v}_\alpha$$

$$m\underbrace{\sum_\alpha n_\alpha \delta\vec{v}_\alpha}_{0} \otimes \vec{v} + m\vec{v}\otimes\underbrace{\sum_\alpha n_\alpha \delta\vec{v}_\alpha}_{0}$$

These are the dynamic stresses due to the bulk motion of the fluid (force exerted by the wind).

This is the stress tensor in the <u>rest frame</u> of the fluid, and it is what is conventionally called the stress tensor $\overleftrightarrow{T}$. $\overleftrightarrow{T}$ is manifestly symmetric, and for an ordinary fluid, which is isotropic, $\overleftrightarrow{T}$ is a multiple of the unit tensor, with the multiple being the pressure p. Generally, $\overleftrightarrow{T}$ is <u>symmetric</u> — otherwise, angular momentum is not conserved — but in more complicated fluids or solids, $\overleftrightarrow{T}$ has off-diagonal terms called <u>shear stresses</u>. Shear stresses correspond to momentum carried not by the particles, but by interactions between particles.

$$\sum_\alpha m n_\alpha \vec{v}_\alpha \otimes \vec{v}_\alpha = \rho \vec{v} \otimes \vec{v} + \overleftrightarrow{T}$$

Unlike the inertia tensor, the stress tensor is a tensor field.

$$\overleftrightarrow{T} = m \sum_\alpha n_\alpha \vec{v}_\alpha \otimes \vec{v}_\alpha \longleftrightarrow m \int d^3\delta v \, n(\vec{r}, \delta\vec{v}) \, \delta\vec{v} \otimes \delta\vec{v}$$

$$= \sum_{j,k} T_{jk}(\vec{r}) \, \hat{e}_j \otimes \hat{e}_k = \sum_{j,k} T_{jk}(\vec{r}) \, |e_j\rangle\langle e_k|$$

$$\longleftrightarrow \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \underset{\text{isotropic fluid}}{=} \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \longleftrightarrow p\overleftrightarrow{I}$$

Fluid equations

ⓐ Mass conservation: $\dfrac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v})$

ⓑ Newton's 2nd Law OR momentum flow

① Integral form:

$$\frac{d}{dt}\int_V \rho\vec{v}\, d\tau = \int_V \underset{\substack{\text{external force density} \\ \text{(source of momentum)}}}{\vec{f}} \, d\tau - \underbrace{\oint_S \rho\vec{v}\otimes\vec{v}\cdot d\vec{a}} - \underbrace{\oint_S \overleftrightarrow{T}\cdot d\vec{a}}$$

$$= \int_V \underbrace{\nabla\cdot\overleftrightarrow{T}}_{\hat{e}_j \frac{\partial T_{jk}}{\partial x^k}} \, d\tau$$

$$= \oint_S \underbrace{\rho\vec{v}(\vec{v}\cdot d\vec{a})}_{\hat{e}_j \rho v_j \vec{v}\cdot d\vec{a}} = \int_V \underbrace{\nabla\cdot(\rho\vec{v}\otimes\vec{v})}_{\hat{e}_j \nabla\cdot(\rho v_j \vec{v}) = \hat{e}_j \frac{\partial}{\partial x^k}(\rho v_j v_k)} \, d\tau$$

If $\overleftrightarrow{T} = p\overleftrightarrow{I}$, then

$$\nabla\cdot\overleftrightarrow{T} = \hat{e}_j T_{jk,k} = \hat{e}_j \delta_{jk} p_{,k} = \hat{e}_j p_{,j} = \nabla p$$

Differential form:

$$\frac{\partial(\rho\vec{v})}{\partial t} = \vec{f} - \underbrace{\nabla\cdot(\rho\vec{v}\otimes\vec{v})} - \nabla\cdot\overleftrightarrow{T}$$

$$\hat{e}_j \nabla\cdot(\rho v_j \vec{v}) = \hat{e}_j \nabla(\rho v^j)\cdot\vec{v} + \hat{e}_j \rho v^j \nabla\cdot\vec{v}$$

$$= \vec{v}\cdot\nabla(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v})$$

$$\frac{\partial(\rho\vec{v})}{\partial t} = \vec{f} - \vec{v}\cdot\nabla(\rho\vec{v}) - \rho\vec{v}(\nabla\cdot\vec{v}) - \nabla p$$

Example: hydrostatic equilibrium $\frac{\partial}{\partial t}=0$, $\vec{v}=0$

$$0=\vec{f}-\nabla\cdot\overleftrightarrow{T}=-\rho g\hat{z}-\nabla p$$

Incompressible (water) vs. compressible (air)

The pressure <u>gradient</u> supports the fluid.

Buoyancy force.

② Integral form:

$$\frac{d}{dt}\int_{V(t)}\rho\vec{v}\,d\tau=\int\vec{f}\,d\tau-\oint_S\overleftrightarrow{T}\cdot\vec{da}$$

comoving volume

Differential form:

$$\frac{d}{dt}(\rho\vec{v}\,\Delta V)=\vec{f}\,\Delta V-\nabla\cdot\overleftrightarrow{T}\,\Delta V$$

$$\frac{d(\rho\vec{v})}{dt}\Delta V+\rho\vec{v}\,\frac{d\Delta V}{dt}=\Delta V\left(\frac{d(\rho\vec{v})}{dt}+\rho\vec{v}\,\nabla\cdot\vec{v}\right)$$

So $\frac{d(\rho\vec{v})}{dt}=\vec{f}-\rho\vec{v}\,\nabla\cdot\vec{v}-\nabla\cdot\overleftrightarrow{T}$

Equivalence: $\frac{d(\rho\vec{v})}{dt}=\frac{\partial(\rho\vec{v})}{\partial t}+\sum_j\frac{\partial(\rho\vec{v})}{\partial x^j}\frac{dx^j}{dt}=\frac{\partial(\rho\vec{v})}{\partial t}+\vec{v}\cdot\nabla(\rho\vec{v})$

So $\frac{\partial(\rho\vec{v})}{\partial t}=\vec{f}-\vec{v}\cdot\nabla(\rho\vec{v})-\rho\vec{v}\,\nabla\cdot\vec{v}-\nabla\cdot\overleftrightarrow{T}$ ✓