Phys 366

Lectures 20-21

Special functions

Factorial function $\quad n! = n(n-1)\cdots 1 = \begin{pmatrix}\text{\# of ways of}\\ \text{ordering } n \text{ objects}\end{pmatrix}$

Binomial coefficient: $\quad \binom{N}{n} = \dfrac{N(N-1)\cdots(N-n+1)}{n!} = \dfrac{N!}{n!\,(N-n)!}$

$= \begin{pmatrix}\text{\# of ways of choosing } n \text{ objects}\\ \text{out of } N \text{ objects, without regard}\\ \text{to order}\end{pmatrix}$

$N=10,\ n=5$

bit sequence	0	0	1	1	1	0	1	0	0	1
objects	1	2	3	4	5	6	7	8	9	10

object chosen if 0 above

$= \begin{pmatrix}\text{\# of bit sequences of length}\\ N \text{ with } n \text{ 0s and } N-n \text{ 1s}\end{pmatrix}$

Binomial theorem $\quad (a+b)^N = \displaystyle\sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n}$

$$2^N = (1+1)^N = \sum_{n=0}^{N}\binom{N}{n} = \begin{pmatrix}\text{\# of bit sequences}\\ \text{of length } N\end{pmatrix}$$

Multinomial coefficient: $\quad \dfrac{N!}{n_1!\cdots n_L!} = \begin{pmatrix}\text{\# of ways of distributing}\\ N \text{ objects to } L \text{ bins with}\\ n_j \text{ objects in bin } j\end{pmatrix}$

$N=12,\ L=4,$
$n_1=n_2=n_3=n_4=3$

sequence	3	2	4	1	2	1	3	2	1	4	4	3
objects	1	2	3	4	5	6	7	8	9	10	11	12

$= \begin{pmatrix}\text{\# of sequences of } L \text{ digits of}\\ \text{length } N\text{, with } n_j \text{ occurrences}\\ \text{of digit } j\end{pmatrix}$

Multinomial expansion $\quad (a_1+a_2+\dots+a_L)^N = \displaystyle\sum_{\substack{n_1,n_2,\dots,n_L\\ n_1+\dots+n_L=N}} \frac{N!}{n_1!\cdots n_L!}\, a_1^{n_1}\cdots a_L^{n_L}$

$$L^N = (1+1+\dots+1)^N = \sum_{\substack{n_1,n_2,\dots,n_L\\ n_1+\dots+n_L=N}} \frac{N!}{n_1!\cdots n_L!} = \begin{pmatrix}\text{\# of sequences}\\ \text{of } L \text{ digits of}\\ \text{length } N\end{pmatrix}$$

Taylor expansions: $\quad f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$

$$f(x_1,\dots,x_L) = \sum_{n_1,\dots,n_L} \frac{1}{n_1!\cdots n_L!} \left.\frac{\partial^{n_1+\dots+n_L} f}{\partial x_1^{n_1}\cdots\partial x_L^{n_L}}\right|_{x_1=\dots=x_L=0} x_1^{n_1}\cdots x_L^{n_L}$$

$$= \sum_{N=0}^{\infty}\frac{1}{N!}\sum_{\substack{n_1,\dots,n_L\\ n_1+\dots+n_L=N}} \frac{N!}{n_1!\cdots n_L!}\left.\frac{\partial^N f}{\partial x_1^{n_1}\cdots\partial x_L^{n_L}}\right|_{x_1=\dots=x_L=0} x_1^{n_1}\cdots x_L^{n_L}$$

Consider the integral $I_n(\alpha) = \int_0^\infty dx\, x^n e^{-\alpha x}$, $n = 0, 1, 2, \ldots$

$$I_0(\alpha) = \frac{1}{\alpha}$$

1. Integrate by parts:

$$n \geq 1: \; I_n(\alpha) = x^n \left(-\frac{e^{-\alpha x}}{\alpha} \right) \Big|_0^\infty - \int_0^\infty dx\, n x^{n-1} \left(-\frac{e^{-\alpha x}}{\alpha} \right)$$

$$I_n(\alpha) = \frac{n}{\alpha} I_{n-1}(\alpha) \longleftarrow \text{recursion relation}$$

Solution: $I_n(\alpha) = \frac{n!}{\alpha^{n+1}}$ Convention: $0! = 1$

2. Differentiate wrt parameter:

$$\frac{dI_n}{d\alpha} = -I_{n+1} \longleftarrow \text{a different recursion relation}$$

Solution: $I_n(\alpha) = \frac{n!}{\alpha^{n+1}}$

3. Generating function: $f(\alpha, \lambda) = \sum_{n=0}^\infty \frac{I_n(\alpha)}{n!} \lambda^n$, $I_n(\alpha) = \frac{\partial^n f}{\partial \lambda^n}\Big|_{\lambda=0}$

$$f(\alpha,\lambda) = \int_0^\infty dx\, e^{(\lambda-\alpha)x} = \frac{1}{\alpha-\lambda}, \quad \alpha > \lambda$$

$$I_n(\lambda) = \frac{\partial^n f}{\partial \lambda^n}\Big|_{\lambda=0} = \frac{n!}{(\alpha-\lambda)^{n+1}}\Big|_{\lambda=0} = \frac{n!}{\alpha^{n+1}}$$

The Gamma function: $\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x}$, $Re(z) > 0$

$$\Gamma(n+1) = n!$$

Recursion relation: $\Gamma(z+1) = \underbrace{-x^z e^{-x}\Big|_0^\infty}_{0 \text{ for } Re(z)>0} - z \int_0^\infty dx\, x^{z-1} e^{-x} = z\Gamma(z)$

$$\boxed{\Gamma(z+1) = z\Gamma(z) \text{ for } Re(z) > 0}$$

We use the recursion relation to extend $\Gamma(z)$ to $Re(z) \leq 0$, but it has singularities at $Re(z) = -n$.

$$\Gamma(1) = \int_0^\infty dx\, e^{-x} = 1 = 0!, \quad \Gamma(n+1) = n!, \quad \Gamma(n) = (n-1)!$$

Another form of the Γ function:

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x} \underset{\uparrow}{=} 2\int_0^\infty dy\, y^{2z-1} e^{-y^2}$$

$$x = y^2$$
$$dx = 2y\,dy$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty dy\, e^{-y^2} = \int_{-\infty}^\infty dy\, e^{-y^2} = \sqrt{\pi}$$

$$n \geq 1: \quad \Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\cdots\tfrac{3}{2}\,\tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n}\sqrt{\pi}$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

Calculating the "area" a_{N-1} of a sphere of <u>unit</u> radius in N dimensions. This is called the unit $(N-1)$-sphere

$$\frac{1}{2}a_{N-1}\Gamma(N/2) = a_{N-1}\int_0^\infty dy\, y^{N-1} e^{-y^2}$$

$$= \int_0^\infty r^{N-1}\,dr\, e^{-r^2} \underbrace{\int d\Omega_{N-1}}_{a_{N-1}}$$

"solid angle" on the $(N-1)$-sphere

$$= \int r^{N-1}\,dr\,d\Omega_{N-1}\, e^{-r^2}$$

$$r^2 = x_1^2 + \cdots + x_N^2$$
$$r^{N-1}\,dr\,d\Omega_{N-1} = dx_1\cdots dx_N$$

$$= \int dx_1\cdots dx_N\, e^{-(x_1^2+\cdots+x_N^2)}$$

$$= \left(\int_{-\infty}^\infty dx\, e^{-x^2}\right)^N$$

$$= \pi^{N/2}$$

$$a_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)} = \begin{cases} (2\pi)^{N/2}/(N-2)!!, & N \text{ even} \\ 2(2\pi)^{(N-1)/2}/(N-2)!!, & N \text{ odd} \end{cases}$$

$N=2$: $a_1 = 2\pi$
$N=3$: $a_2 = 4\pi$
$N=4$: $a_3 = 2\pi^2$
$N=5$: $a_4 = 8\pi^2/3$

$$N \text{ even}: \quad \Gamma\left(\tfrac{N}{2}\right) = \left(\tfrac{N}{2}-1\right)! = \underbrace{\left(\tfrac{N}{2}-1\right)\left(\tfrac{N}{2}-2\right)\cdots 2\cdot 1}_{N/2-1 \text{ terms}} = \frac{(N-2)!!}{2^{N/2-1}}$$

$$N \text{ odd}: \quad \Gamma\left(\tfrac{N}{2}\right) = \Gamma\left(\tfrac{N-1}{2}+\tfrac{1}{2}\right) = \frac{(N-2)!!}{2^{(N-1)/2}}\sqrt{\pi}$$

Volume interior to (N-1)-sphere of radius R:

$$\mathcal{V}_N = \int_0^R r^{N-1}\,dr\,d\Omega_{N-1} = a_{N-1}\int_0^R dr\, r^{N-1} = \frac{1}{N} R^N a_{N-1}$$

$$\mathcal{V}_N = \frac{1}{N} R^N a_{N-1}\,, \qquad \frac{d\mathcal{V}_N}{dR} = R^{N-1} a_{N-1}$$

$N=2$: $\mathcal{V}_2 = \pi R^2$
$N=3$: $\mathcal{V}_3 = 4\pi R^3/3$
$N=4$: $\mathcal{V}_4 = \pi^2 R^4/2$
$N=5$: $\mathcal{V}_5 = 8\pi^2 R^5/15$

Asymptotic Series

Boas has a very nice discussion of the error function and how to get its asymptotic series from integration by parts

$$f(x) = \sum_{n=0}^{N} \frac{a_n}{x^n} + R_N(x)$$

The series might not converge as $N\to\infty$, but the finite sum would still be a good approximation as long as $|R_N(x)|$ is small compared to $|f(x)|$ and $\left|\sum_{n=0}^{N} a_n/x^n\right|$.
Asymptotic series capture this notion.

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$ if (for all N) the error incurred

↑ is asymptotic to

in terminating the series at $n=N$ is small compared to $1/|z|^N$; precisely, the error goes to zero faster than $1/|z|^N$ as $|z|\to\infty$ (for some range of arg z), i.e.,

$$\left| f(z) - \sum_{n=0}^{N} a_n/z^n \right| |z|^N \xrightarrow[|z|\to\infty]{} 0 \qquad \text{for all } N$$

This is sometimes written as

$$f(z) = \sum_{n=0}^{N} \frac{a_n}{z^n} + O\left(\frac{1}{|z|^{N+1}}\right)$$

Ordinary convergence: $f(z) \xrightarrow[N\to\infty]{} \sum_{n=0}^{N} \frac{a_n}{z^N}$

z fixed, $N \to \infty$

Asymptotic convergence: $f(z) \xrightarrow[|z|\to\infty]{} \sum_{n=0}^{N} \frac{a_n}{z^n}$

N fixed, $|z| \to \infty$

More generally,

$$f(z) \sim \sum_{n=0}^{\infty} \phi_n(z) \quad \text{if} \quad \frac{\left|f(z) - \sum_{n=0}^{N} \phi_n(z)\right|}{|\phi_N(z)|} \xrightarrow[|z|\to\infty]{} 0 \quad \text{for all } N$$

<u>Stirling's formula</u>

$$n! = \underbrace{\sqrt{2\pi n}\, n^n e^{-n}}_{\text{1st term in asymptotic series}} \underbrace{e^{\theta/12n}}_{\text{we have control of the error}} \qquad \begin{array}{l} n \geq 0 \\ 0 < \theta < 1 \end{array}$$

$$\underline{\ln n! = n \ln n - n + \tfrac{1}{2}\ln(2\pi n) + \frac{\theta}{12n}}$$

$$\Gamma(z+1) \sim \underbrace{z^z e^{-z}\sqrt{2\pi z}}\left(1 + \underbrace{\frac{1}{12z} + \frac{1}{288z^2} + \cdots}_{}\right)$$

further terms in the asymptotic series can be gotten from the recursion relation $\Gamma(z+1) = z\Gamma(z)$, which is integration by parts.

Where does this come from?

$$\Gamma(z+1) = \int_0^\infty dx\, x^z e^{-x}$$

When z is large, $x^z e^{-x}$ is highly peaked at $x = z$

Look at $\ln(x^z e^{-x}) = z \ln x - x$, which varies slowly near the peak, so a few terms in a Taylor expansion should provide a good approximation.

What we are doing is a baby version of saddle-point integration or the stationary-phase approximation.

$$\frac{d}{dx}(z\ln x - x) = \frac{z}{x} - 1$$

$$= 0 \quad \text{at } x = z$$

$$\frac{d^2}{dx^2}(z\ln x - x) = -\frac{z}{x^2}$$

$$= -\frac{1}{z} \quad \text{at } x = z$$

So, expanding about the peak at $x = z$,

$$\ln(x^z e^{-x}) = z\ln x - x = z\ln z - z - \frac{(x-z)^2}{2z} + \cdots$$

Near the peak,

$$x^z e^{-x} \simeq z^z e^{-z} \exp\left(-\frac{(x-z)^2}{2z}\right)$$

We would never have found a Gaussian peak had we not expanded the logarithm.

So

$$\Gamma(z+1) = \int_0^\infty dx\, x^z e^{-x}$$

$$\sim z^z e^{-z} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-(x-z)^2/2z}}_{\sqrt{2\pi z}}$$

extend to $-\infty$

$$\Gamma(z+1) \sim z^z e^{-z} \sqrt{2\pi z}$$