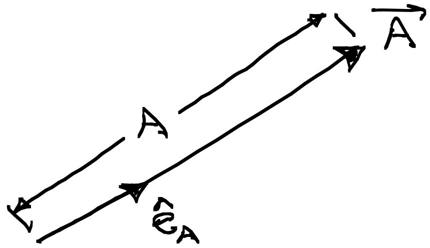
Phys 366

Lectures 3-4

Vector algebra

Vectors

Pictorial view: a vector is a magnitude (length) and a direction.



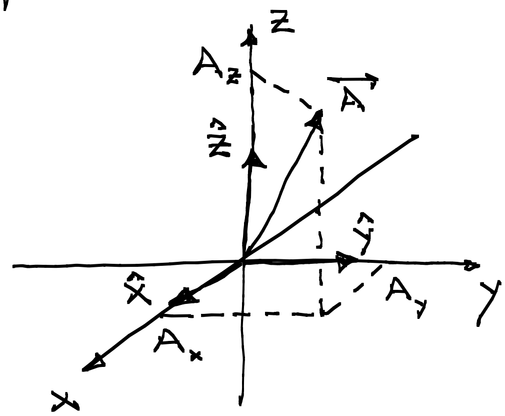
$$\mathbf{A} = \vec{A} = A\,\hat{e}_A$$

A ← magnitude $|\vec{A}| = A$

$\hat{e}_A$ ← direction (unit vector)

__Always__ use the symbol (boldface, arrow, hat) that distinguishes vectors from numbers.

Components and basis vectors:



Cartesian basis vectors

$\hat{i}$ $\hat{j}$ $\hat{k}$ ← Worst choice. Boas uses. We won't.

$\hat{x}$ $\hat{y}$ $\hat{z}$

$\hat{e}_x$ $\hat{e}_y$ $\hat{e}_z$

$\hat{e}_1$ $\hat{e}_2$ $\hat{e}_3$

__orthonormal__

vectors

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} = \sum_j A_j\hat{e}_j = A_j\hat{e}_j$$

numbers, not vectors; components of a vector

$j = x, y, z$
$j = 1, 2, 3$

Summation convention (sum over repeated indices)

This is a 3-dimensional vector space over the real numbers:

① Zero vector $\vec{0} = 0$.

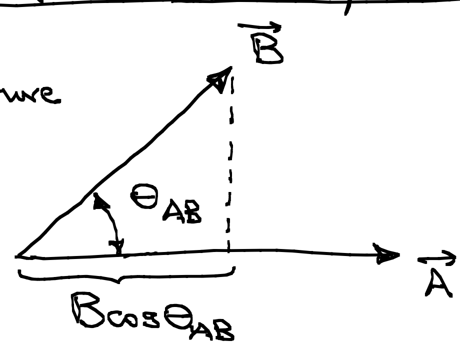
② Addition (commutative and associative)

③ Scalar multiplication (distributive)

Pictures
Components

__Dot (scalar, inner) product__: $\vec{A}\cdot\vec{B} = $ (scalar)

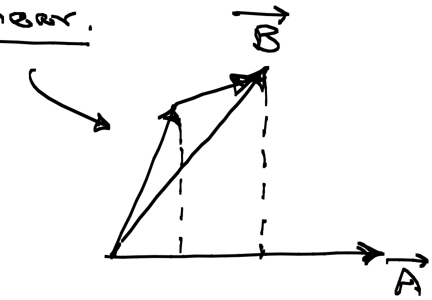
Picture



$$\vec{A}\cdot\vec{B} = AB\cos\theta_{AB}$$

doesn't matter whether we take the bigger or smaller angle between $\vec{A}$ and $\vec{B}$

The dot product is commutative and <u>bilinear</u>.



Components

$$\vec A = \sum_j A_j \hat e_j, \quad A_j = \hat e_j \cdot \vec A$$

The dot product is the <u>component-taking machine.</u>

Orthonormality

$$\hat e_j \cdot \hat e_k = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

Kronecker delta

$$\vec A \cdot \vec B = \Big(\sum_j A_j \hat e_j\Big)\cdot\Big(\sum_k B_k \hat e_k\Big) = \sum_{j,k} A_j B_k \underbrace{\hat e_j \cdot \hat e_k}_{\delta_{jk}} = \sum_j A_j B_j$$

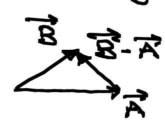
The scalar product is commutative, bilinear, and

$$\vec A \cdot \vec A = \sum_j A_j^2 = |\vec A|^2 = A^2 \geq 0, \quad = 0 \Leftrightarrow \vec A = 0$$

Generalization of Pythagoras. The dot product is the way we calculate lengths.

$$\vec A \cdot \vec B = 0 \Leftrightarrow \vec A \perp \vec B$$
$$>$$
$$<$$

Lots of trig is best done with vector algebra:



$$\begin{aligned} |\vec B - \vec A|^2 &= (\vec B - \vec A)\cdot(\vec B - \vec A) \\ &= \vec B \cdot \vec B + \vec A \cdot \vec A - 2\vec B \cdot \vec A \\ &= B^2 + A^2 - 2BA\cos\theta_{AB} \end{aligned}$$

Law of cosines

Digression:

$$\vec A = \sum_j \hat e_j (\hat e_j \cdot \vec A) = \underbrace{\Big(\sum_j \hat e_j \otimes \hat e_j\Big)}_{\text{unit 2-tensor}} \vec A$$

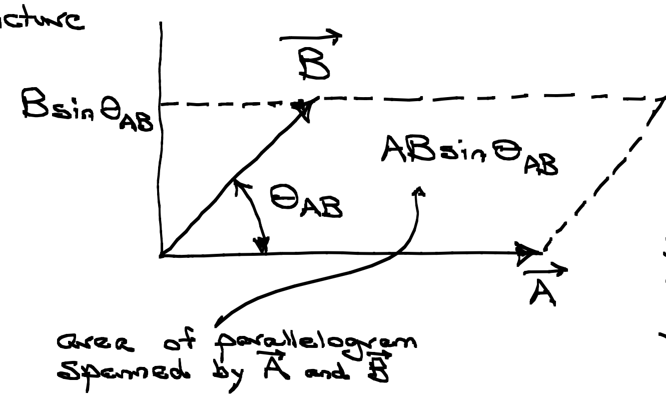
A 2-tensor is a machine that acts linearly on a vector either on the left or right and leaves a vector.

$\otimes$ is the outer or direct or tensor product

We need to get away from the notion of a vector as pointing from one point to another. This only applies to position vectors, which are special to flat spaces; position vectors don't even exist in curved spaces (think of the surface of a sphere).

Cross (vector) product

Picture



special to 3 dimensions

$$\vec{A}\times\vec{B} = \hat{n}\,AB\sin\theta_{AB}$$

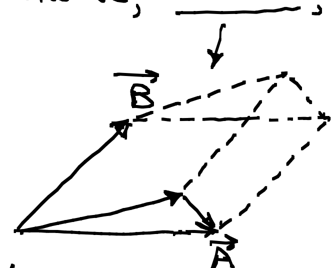
We have to use the smaller angle

$\hat{n}$ is the unit vector out of the plane; its direction is determined by the right-hand rule.

The cross product is the generator of oriented surface areas in 3 dimensions.

The cross product is anticommutative, not associative, bilinear, and

$$\vec{A}\times\vec{B} = 0 \Longleftrightarrow \vec{A}\parallel\vec{B}$$



Components: $\hat{e}_z\cdot\hat{e}_x\times\hat{e}_y = \hat{e}_x\cdot\hat{e}_y\times\hat{e}_z = \hat{e}_y\cdot\hat{e}_z\times\hat{e}_x = +1$

$$\hat{e}_z\cdot\hat{e}_y\times\hat{e}_x = \hat{e}_x\cdot\hat{e}_z\times\hat{e}_y = \hat{e}_y\cdot\hat{e}_x\times\hat{e}_z = -1$$

$$\hat{e}_j\cdot\hat{e}_k\times\hat{e}_l = \epsilon_{jkl} = \begin{cases} +1 & \text{if } jkl \text{ is an even permutation of } 123 \\ -1 & \text{if } jkl \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases}$$

right-handed coördinate system

antisymmetric symbol
Levi-Civita tensor

$$\vec{A}\times\vec{B} = \sum_{j,k} A_j\hat{e}_j\times B_k\hat{e}_k = \sum_{j,k}\epsilon_{jkl}A_jB_k\hat{e}_l = \sum_{j,k}\hat{e}_l\,\epsilon_{ljk}A_jB_k$$

Explicitly, $\vec{A}\times\vec{B} = \hat{e}_x(A_yB_z - A_zB_y) + \hat{e}_y(A_zB_x - A_xB_z) + \hat{e}_z(A_xB_y - A_yB_x)$

Digression: Antisymmetric symbol

$$\hat{e}_j\cdot\hat{e}_k\times\hat{e}_l = \epsilon_{jkl} = \begin{cases} +1 & \text{if } jkl \text{ is an even permutation of } 123 \\ -1 & \text{if } jkl \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases}$$

antisymmetric symbol
Levi-Civita tensor

Antisymmetry: ϵ_{jkl} is antisymmetric under exchange of any two indices, e.g., $\epsilon_{kjl} = -\epsilon_{jkl}$.

Cyclic property: $\epsilon_{jkl} = \epsilon_{klj} = \epsilon_{ljk}$

Determinant: $\det M = \epsilon_{jkl} M_{1j} M_{2k} M_{3l}$

$$\epsilon_{mnp} \det M = \epsilon_{jkl} M_{mj} M_{nk} M_{pl}$$

$$\Rightarrow \vec{A} \times \vec{B} = \det \begin{pmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix}$$

$$\epsilon_{jkn}\epsilon_{lmn} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

$$\epsilon_{jlm}\epsilon_{klm} = \delta_{jk}\delta_{ll} - \delta_{jl}\delta_{lk} = 2\delta_{jk}$$

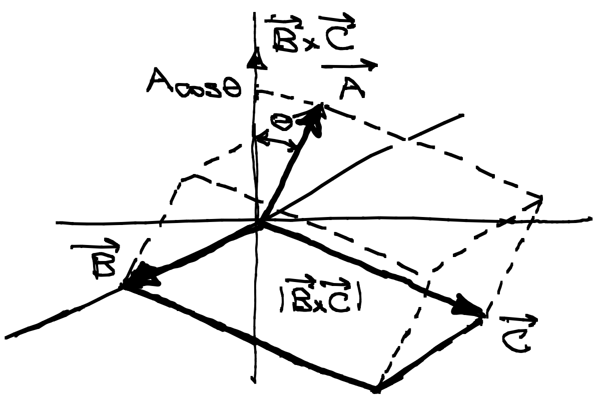
Scalar triple product: $\vec{A}\cdot\vec{B}\times\vec{C} = \left(\begin{array}{c}\underline{\text{oriented}} \text{ volume of parallepiped} \\ \text{spanned by } \vec{A}, \vec{B}, \text{ and } \vec{C}\end{array}\right)$

$$= \sum_{j,k,l} \epsilon_{jkl} A_j B_k C_l$$

$$= \det \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix}$$

$$= \vec{B}\cdot\vec{C}\times\vec{A} = \vec{C}\cdot\vec{A}\times\vec{B}$$

cyclic property



volume of parallelepiped is

$$\underbrace{|\vec{B}\times\vec{C}|}_{\text{area of base}} \underbrace{A\cos\theta}_{\text{height}} = |\vec{A}\cdot\vec{B}\times\vec{C}|$$

Vector triple product:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \hat{e}_j \epsilon_{jkl} A_k \epsilon_{lmn} B_m C_n = \hat{e}_j A_k B_m C_n \underbrace{\epsilon_{jkl}\epsilon_{lmn}}$$

$$= \epsilon_{ljk}\epsilon_{lmn} \quad \text{cyclic property}$$

$$= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$\neq (\vec{A}\times\vec{B})\times\vec{C}$

not associative

$$= \hat{e}_j B_j A_k C_k - \hat{e}_j C_j A_k B_k$$

$$= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$ BAC-CAB rule

Let's do $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = \vec{C} \cdot [\vec{D} \times (\vec{A} \times \vec{B})]$ cyclic property

$$= \vec{C} \cdot [\vec{A}(\vec{D} \cdot \vec{B}) - \vec{B}(\vec{D} \cdot \vec{A})]$$

$$= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$