Phys 366

Lectures 7-8

Vector fields, gradient, and divergence

Fields

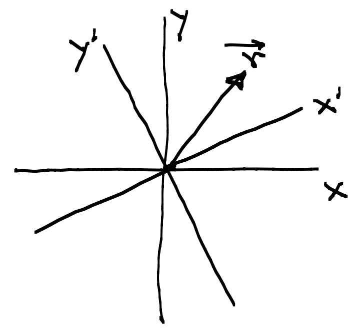
Scalar fields: pressure $p(\vec{r})$

temperature $T(\vec{r})$

number density $n(\vec{r})$

mass density $\rho(\vec{r})$

probability density $P_r(\vec{r})$



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$
$$= x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$$

$$p(x,y,z) = p(x',y',z')$$

pseudoscalar?

Vector fields: force field $\vec{F}(\vec{r}) = m\vec{g}(\vec{r})$

gravitational force ... gravitational field

$$\vec{F}(\vec{r}) = q\,\vec{E}(\vec{r}) + \vec{v}\times\vec{B}(\vec{r})$$

Lorentz force ... electric field ... magnetic field

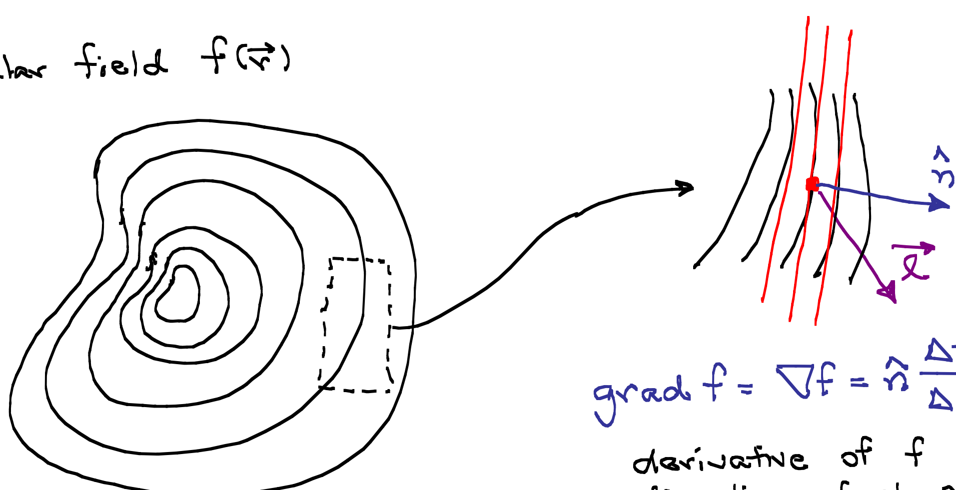
fluid velocity $\vec{v}(\vec{r})$

$$v_j(x,y,z) = \hat{e}_j \cdot \vec{v}(\vec{r})$$
$$= \sum_k \hat{e}_j\cdot\hat{e}'_k \; \hat{e}'_k\cdot\vec{v}(\vec{r})$$
$$= \sum_k \hat{e}_j\cdot\hat{e}'_k \; v'_k(x',y',z')$$

pseudovector?

Gradient: local (linear) approximation to level structure (contours) of a scalar function f.

Scalar field $f(\vec{r})$



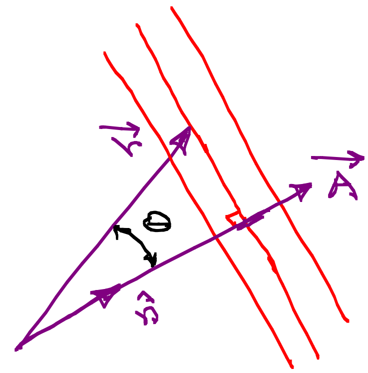
$$\Delta s = \Delta l\, \hat{n}\cdot\hat{\ell}$$

$$\text{grad } f = \nabla f = \hat{n}\frac{\Delta f}{\Delta s} = \hat{n}\frac{df}{ds}$$

derivative of f along direction of steepest ascent

Examples:

① Linear function



$$f(\vec{r}) = \vec{A}\cdot\vec{r} = A\hat{n}\cdot\vec{r}$$

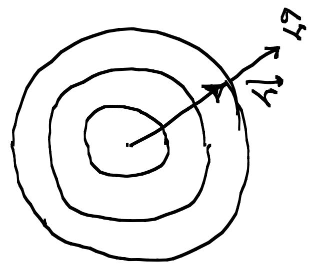
$$\nabla f = \frac{df}{ds}\hat{n} = A\hat{n} = \vec{A}$$

$$\hat{n}\cdot\vec{r} = r\cos\theta = s$$

② Spherically symmetric function

$$f(\vec{r}) = f(r)$$

$$\nabla f = \frac{df}{dr}\hat{r}$$



$$\frac{df}{ds} = \frac{df}{dr}$$

Directional derivative

$$\frac{\partial f}{\partial l} = \frac{\Delta f}{\Delta l} = \hat{l}\cdot\hat{n}\frac{\Delta f}{\Delta s} = \hat{l}\cdot\nabla f$$

Cartesian components

$$\frac{\partial f}{\partial x} = \hat{x}\cdot\nabla f, \quad \frac{\partial f}{\partial y} = \hat{y}\cdot\nabla f, \quad \frac{\partial f}{\partial z} = \hat{z}\cdot\nabla f, \quad \frac{\partial f}{\partial x_j} = \nabla f\cdot\hat{e}_j$$

$$\nabla f = \sum_j \hat{e}_j(\hat{e}_j\cdot\nabla f) = \sum_j \frac{\partial f}{\partial x_j}\hat{e}_j$$

∇ as an operator: $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \sum_j \hat{e}_j\frac{\partial}{\partial x_j}$

Curvilinear coördinates:

① Cylindrical: $\frac{\partial f}{\partial \rho} = \hat{\rho}\cdot\nabla f \qquad \frac{1}{\rho}\frac{\partial f}{\partial \phi} = \hat{\phi}\cdot\nabla f, \qquad \frac{\partial f}{\partial z} = \hat{z}\cdot\nabla f$

$l = \rho\phi$

$$\nabla f = \hat{\rho}\frac{\partial f}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{z}\frac{\partial f}{\partial z}$$

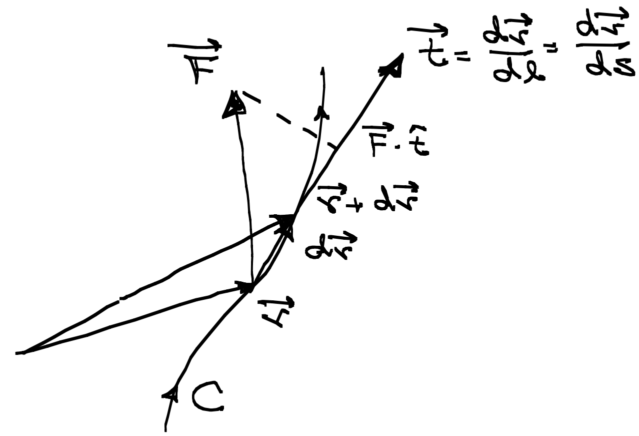
Spherical: $\dfrac{\partial f}{\partial r} = \hat{r}\cdot\nabla f$, $\dfrac{1}{r}\dfrac{\partial f}{\partial \theta} = \hat{\theta}\cdot\nabla f$, $\dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi} = \hat{\phi}\cdot\nabla f$

$\ell = r\theta$ $\qquad \ell = r\sin\theta\,\phi = \rho\phi$

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Integral representation

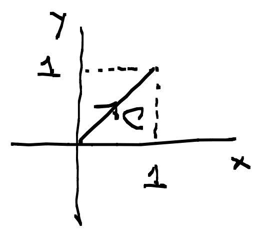
Line integrals



$$\int_C \vec{F}\cdot d\vec{\ell} = \int_{\ell_i}^{\ell_f} \vec{F}\cdot\frac{d\vec{r}}{d\ell}\,d\ell$$

$d\vec{\ell}$: $d\vec{r}$ or $d\vec{s}$

could be work transporting a particle along C

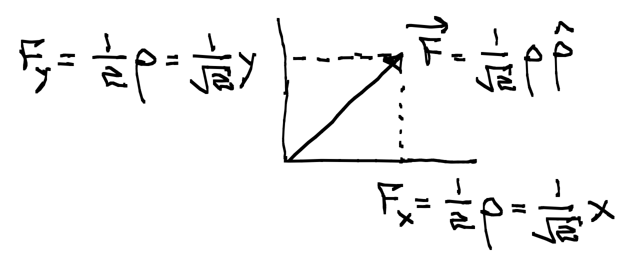


Example: $\vec{F}(\vec{r}) = \hat{\rho}\,\rho\cos\phi = x\,\hat{\rho}$

$$\int_C \underbrace{\vec{F}\cdot d\vec{r}}_{d\vec{r}\,=\,\hat{\rho}\,d\rho} = \int_0^{\sqrt{2}} d\rho\,\rho\underbrace{\cos\phi}_{=1/\sqrt{2}} = \frac{1}{\sqrt{2}}\underbrace{\int_0^{\sqrt{2}} d\rho\,\rho}_{\frac{1}{2}\rho^2\big|_0^{\sqrt{2}} = 1} = \frac{1}{\sqrt{2}}$$

$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 dx\,F_x + \int_0^1 dy\,F_y$$



$$= \frac{1}{\sqrt{2}}\underbrace{\int_0^1 x\,dx}_{1/2} + \frac{1}{\sqrt{2}}\underbrace{\int_0^1 y\,dy}_{1/2}$$

$$= \frac{1}{\sqrt{2}}$$

Path dependence? Closed curves? Stokes theorem?

Work-energy theorem:

$$\begin{pmatrix}\text{work done by force} \\ \vec{F} \text{ along curve } C\end{pmatrix} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C \vec{F}\cdot \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}}\, dt$$

we are not using arclength to parametrize the curve

$$= \int_C \vec{F}\cdot\vec{v}\, dt$$

$$= \int_C m\dot{\vec{v}}\cdot\vec{v}\, dt$$

$$= \int_C \frac{d}{dt}\underbrace{\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right)}_{T = (\text{kinetic energy})} dt$$

$$= \int_C dT$$

$$= \Delta T$$

Newton's gravity

$$\vec{F} = m\vec{g} = -m\nabla\Phi$$

gravitational potential

If the force is conservative, $\vec{F} = -\nabla V$

potential energy

Electrostatics

$$\vec{F} = q\vec{E} = -q\nabla\Phi$$

electrostatic potential

Magnetic force $\vec{F}$.

$$\int_C \vec{F}\cdot d\vec{r} = -\int_C \nabla V\cdot d\vec{l}$$

$$= -\int_{l_i}^{l_f} \frac{\partial V}{\partial l}\, dl$$

$$= -(V_f - V_i)$$

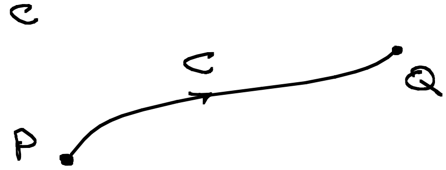
$$= -\Delta V$$

$$\Longrightarrow \Delta(T+V) = 0$$

Conservation of mechanical energy

Fundamental theorem of calculus:

$$\int_C \nabla f\cdot d\vec{l} = f(Q) - f(P) \qquad \oint_C \nabla f\cdot d\vec{l} = 0$$



Is ∇f a vector? Passive transformation of position vector

$$x'_j = \hat{e}'_j \cdot \vec{r} = \underbrace{\hat{e}'_j \cdot \hat{e}_k}_{M_{jk}} x_k \,, \qquad \hat{e}'_j = \hat{e}_k \underbrace{(\hat{e}_k \cdot \hat{e}'_j)}_{M_{jk}}$$

$$x_k = \hat{e}_k \cdot \vec{r} = \underbrace{\hat{e}_k \cdot \hat{e}'_j}_{M_{jk}} x'_j \,, \qquad \hat{e}_k = \hat{e}'_j \underbrace{(\hat{e}'_j \cdot \hat{e}_k)}_{M_{jk}}$$

$$\frac{\partial}{\partial x'_j} = \frac{\partial}{\partial x_k} \frac{\partial x_k}{\partial x'_j} = M_{jk} \frac{\partial}{\partial x^k}$$

These derivatives transform like vectors, so ∇f is a vector. More to the point, ∇ transforms like a vector operator.

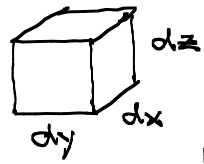
Divergence

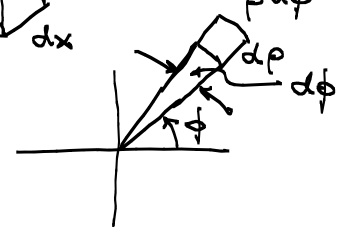
Vector field $\vec{F}(\vec{r})$ $\quad$ $\operatorname{div}\vec{F}$ or $\nabla\cdot\vec{F}$

We need to do a little work on volume and surface integrals first.

Volume integrals $\quad \displaystyle\int_V f(\vec{r})\, dV$ $\leftarrow$ d^3r, $d\tau$ $\leftarrow$ textbook

$f(x,y,z)\, dx\, dy\, dz$



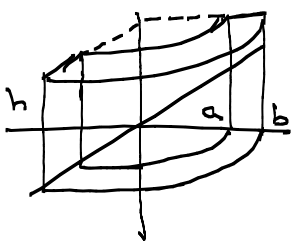


$$f(\rho,\phi,z)\, \underbrace{d\rho\ \rho d\phi\ dz}_{\rho\, d\rho\, d\phi\, dz}$$

$$f(r,\theta,\phi)\, \underbrace{dr\ r d\theta\ \overbrace{r\sin\theta}^{\rho}\, d\phi}$$

$$r^2\sin\theta\, dr\, d\theta\, d\phi = r^2\, dr\, \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega}$$

solid angle (steradians)

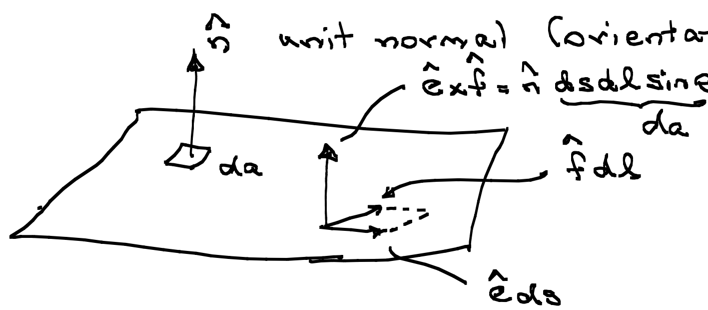


$$\int_a^b \rho\, d\rho \int_0^{\pi/2} d\phi \int_0^h dz\ f(\rho,\phi,z)$$

Surface integrals

$$\int_S \vec{F}\cdot d\vec{a}$$

$\hat{n}\, da = \hat{n}\, d\sigma$ $\leftarrow$ textbook



Cartesian and curvilinear

Closed surfaces (orientation)

unique?

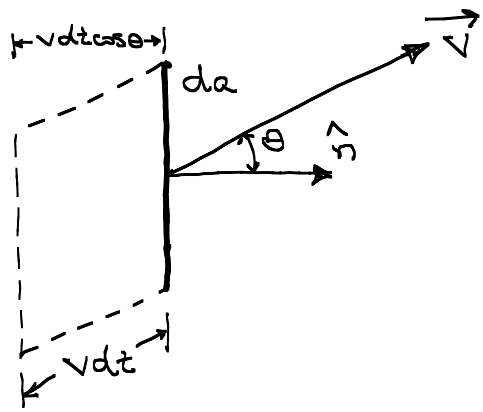
A volume is bounded by a <u>closed</u> surface (orientation?)
The closed surface has no boundary

An open surface is bounded by a <u>closed</u> curve (orientation?)
The closed curve has no boundary.

An open curve is bounded by two points. The points have no boundary.

(Flux of $\vec{F}$ through S) $= \int_S \vec{F}\cdot\vec{da}$. This comes from transport theory:

Particles with number density $n(\vec{r})$ and velocity $\vec{v}(\vec{r})$.



$$\begin{pmatrix}\text{number of particles that} \\ \text{pass through } \vec{da} \text{ in time } dt\end{pmatrix} = n\, da\, v\, dt \cos\theta = n\vec{v}\cdot\vec{da}\, dt$$

$$(\text{flux of particles through } \vec{da}) = n\vec{v}\cdot\vec{da}$$

Many sorts of particles: $n_\alpha, \vec{v}_\alpha$

$$(\text{flux of particles through } \vec{da}) = \underbrace{\sum_\alpha n_\alpha \vec{v}_\alpha}_{n\langle\vec{v}\rangle}\cdot\vec{da}$$

$$n = \sum_\alpha n_\alpha$$

$$\langle\vec{v}\rangle = \frac{1}{n}\sum_j n_\alpha \vec{v}_\alpha$$

$$(\text{flux of particles through } S) = \int_S n\langle\vec{v}\rangle\cdot\vec{da}$$

Particles carrying some property, e.g., charge: $q, n_\alpha, \vec{v}_\alpha$

$$(\text{flux of charge through } \vec{da}) = \underbrace{\sum_\alpha q\, n_\alpha \vec{v}_\alpha}\cdot \vec{da}$$

$\parallel$

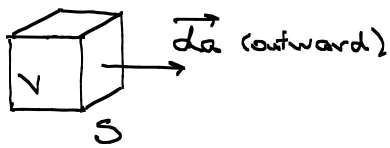
(current through $\vec{da}$)

$$\equiv \vec{J} = (\text{current density}) = qn\langle\vec{v}\rangle$$

$$\begin{pmatrix}\text{charge}\\ \text{density}\end{pmatrix} = \rho = qn = q\sum_\alpha n_\alpha$$

$$(\text{current through } S) = \int_S \vec{J}\cdot\vec{da}$$

Divergence of a vector field



Shape of volume?

$$\operatorname{div}\vec{F}\, dV = \nabla\cdot\vec{F}\, dV \equiv \oint_S \vec{F}\cdot\vec{da}$$

$\nabla\cdot\vec{F}$ is the differential version of outward flux.

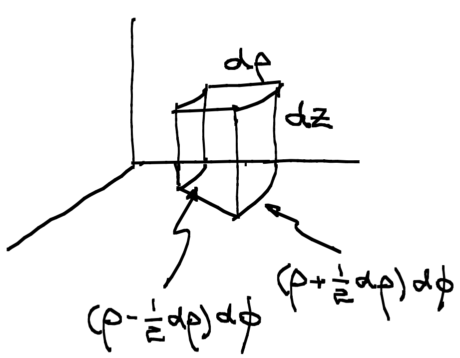
Components

Cartesian:
$$\oint_S \vec{F}\cdot\vec{da} = \sum_j \underbrace{F_j(x_j+\tfrac{1}{2}dx_j)\,da_j - F_j(x_j-\tfrac{1}{2}dx_j)\,da_j}_{\frac{\partial F_j}{\partial x_j}dx_j\,da_j} = dV\sum_j \frac{\partial F_j}{\partial x_j}$$

$$\nabla\cdot\vec{F} = \sum_j \frac{\partial F_j}{\partial x^j} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

∇ operator

Cylindrical:
$$\oint_S \vec{F}\cdot\vec{da} = \underbrace{F_\rho(\rho+\tfrac{1}{2}d\rho)(\rho+\tfrac{1}{2}d\rho)\,d\phi\, dz - F_\rho(\rho-\tfrac{1}{2}d\rho)(\rho-\tfrac{1}{2}d\rho)\,d\phi\, dz}_{\frac{\partial(\rho F_\rho)}{\partial\rho}d\rho\, d\phi\, dz = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho}dV}$$



$$+ \underbrace{F_\phi(\phi+\tfrac{1}{2}d\phi)\,d\rho\, dz - F_\phi(\phi-\tfrac{1}{2}d\phi)\,d\rho\, dz}_{\frac{\partial F_\phi}{\partial\phi}d\rho\, d\phi\, dz = \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}dV}$$

$$+ \underbrace{F_z(z+\tfrac{1}{2}dz)\,d\rho\,\rho\, d\phi - F_z(z-\tfrac{1}{2}dz)\,d\rho\,\rho\, d\phi}_{\frac{\partial F_z}{\partial z}\rho\, d\rho\, d\phi\, dz = \frac{\partial F_z}{\partial z}dV}$$

$$\nabla\cdot\vec{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{1}{\rho}\frac{\partial(\rho F_z)}{\partial z}$$

	↑	↑	↑
area ⊥ to differentiation	$\rho\, d\phi\, dz$	$d\rho\, dz$	$d\rho\, \rho\, d\phi$
length along differentiation	$d\rho$	$\rho\, d\phi$	dz

$$\nabla\cdot\vec{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}$$

General orthogonal curvilinear coördinates: The derivation for cylindrical coördinates is enough to teach us how to get the divergence for any orthogonal coördinates. Suppose we have orthogonal coördinates x_1, x_2, x_3, with orthonormal basis vectors $\hat{x}_1, \hat{x}_2, \hat{x}_3$. Length along x_j is given by $h_j\,dx_j$, i.e., $h_1\,dx_1$, $h_2\,dx_2$, and $h_3\,dx_3$, so the volume element is $dV = h_1 h_2 h_3\, dx_1\, dx_2\, dx_3$. The two effects, for the derivative with respect to one of the coördinates, are (i) the change in the area element from the front to the back of dV due to the other two coördinates and (ii) the conversion of the derivative to length. So we get

$$\nabla\cdot\vec{F} = \frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3F_1)}{\partial x_1} + \frac{1}{h_2h_1h_3}\frac{\partial(h_1h_3F_2)}{\partial x_2} + \frac{1}{h_3h_1h_2}\frac{\partial(h_1h_2F_3)}{\partial x_3}$$

	↑	↑	↑
area ⊥ to differentiation	$h_2h_3\,dx_2\,dx_3$	$h_1h_3\,dx_1\,dx_3$	$h_1h_2\,dx_1\,dx_2$
length along differentiation	$h_1\,dx_1$	$h_2\,dx_2$	$h_3\,dx_3$

<u>OR</u> $$\nabla\cdot\vec{F} = \sum_j \frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_j}\left(\frac{h_1h_2h_3}{h_j}F_j\right)$$

Spherical: $$\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

area ⊥ to differentiation	$r\,d\theta\; r\sin\theta\,d\phi$	$dr\; r\sin\theta\,d\phi$	$dr\; r\,d\theta$
length along differentiation	dr	$r\,d\theta$	$r\sin\theta\,d\phi$

Divergence theorem: $\displaystyle\int_V \nabla\cdot\vec{F}\,dV = \oint_S \vec{F}\cdot\vec{da}$ Gauss's Law

$S \uparrow$ surface that <u>encloses</u> V

Conservation laws

$$N(t) = \begin{pmatrix}\text{number of people}\\ \text{in room at time } t\end{pmatrix} = \int_V n(\vec{r},t)\,dV$$

$$\begin{pmatrix}\text{change in number of people}\\ \text{in room during interval } dt\end{pmatrix} = d\int_V n(\vec{r},t)\,dV = \int_V \frac{\partial n}{\partial t}\,dV$$

$$\| $$

$$\underbrace{\begin{pmatrix}\text{number of people that come}\\ \text{into room during } dt\end{pmatrix} - \begin{pmatrix}\text{number of people that}\\ \text{leave room during } dt\end{pmatrix}}_{-\oint_S n\vec{v}\cdot\vec{da}}$$

$\vec{da}$ ← outward directed

$$+ \underbrace{\begin{pmatrix}\text{number of people born}\\ \text{in room during } dt\end{pmatrix}}_{\text{source}} - \underbrace{\begin{pmatrix}\text{number of people who}\\ \text{died in room during } dt\end{pmatrix}}_{\text{sink}}$$

$= 0$ for a conserved quantity

Conservation law: (integral form)

$$\int_V \frac{\partial n}{\partial t}\,dV = -\oint_S n\vec{v}\cdot\vec{da} = -\int_V \nabla\cdot(n\vec{v})\,dV$$

differential form: $\displaystyle\frac{\partial n}{\partial t} = -\nabla\cdot(n\vec{v})$ <u>OR</u> $\displaystyle\frac{dn}{dt} = -n\nabla\cdot\vec{v}$

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \nabla n\cdot\frac{d\vec{r}}{dt} = \frac{\partial n}{\partial t} + \vec{v}\cdot\nabla n$$

total time derivative

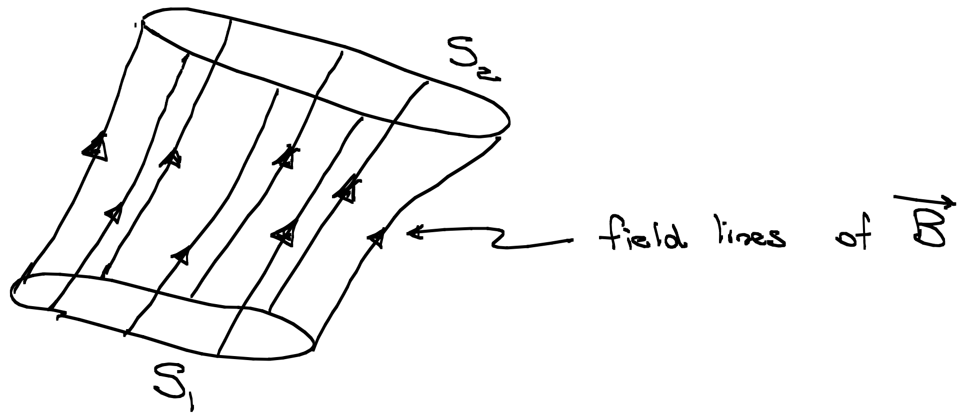
derivative following the flow

$\nabla\cdot\vec{v}$ is the fractional expansion rate. $\nabla\cdot\vec{v}=0$ means <u>incompressible</u> flow, which implies n is a constant along the flow.

$$\frac{d(n\Delta V)}{dt} = \frac{dn}{dt}\Delta V + n\frac{d\Delta V}{dt} = \Delta V\left[\frac{dn}{dt} + n\underbrace{\frac{1}{\Delta V}\frac{d\Delta V}{dt}}_{\nabla\cdot\vec{v}}\right]$$

Field lines and conservation of flux

$\nabla \cdot \vec{B} = 0$



$$0 = \int_V \nabla \cdot \vec{B}\, dV = \int_{S_2} \vec{B} \cdot \vec{da}_2 - \int_{S_1} \vec{B} \cdot \vec{da}_1 + \underbrace{\int_{\text{sides}} \vec{B} \cdot \vec{da}}_{0 \text{ since } \vec{B} \text{ is tangent to the sides}}$$

upward   upward

Flux of a divergenceless field is conserved:

$$\int_{S_2} \vec{B} \cdot \vec{da} = \int_{S_1} \vec{B} \cdot \vec{da}$$

This is what we have in mind when we draw field lines: as the field lines get closer together (further apart), the field gets stronger (weaker).

Is $\nabla \cdot \vec{F}$ a scalar?

$$\frac{\partial F_j'}{\partial x_j'} = M_{jk} \frac{\partial}{\partial x^k} (M_{jl} F_l) = \underbrace{(M^{-1})_{kj} M_{jl}}_{\delta_{kl}} \frac{\partial F_l}{\partial x^k} = \frac{\partial F_k}{\partial x^k}$$

What this does is to use the transformation of ∇ as a vector operator to show $\nabla \cdot \vec{F}$ is a scalar.

Laplacian: $\nabla^2 f = \nabla \cdot \nabla f$ $\qquad$ $\nabla^2 f$ is a scalar field

Cartesian coördinates: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

General <u>orthogonal</u> curvilinear coördinates:

Lengths $h_1 dx_1 \quad h_2 dx_2 \quad h_3 dx_3$

$$\nabla f = \sum_j \frac{1}{h_j} \frac{\partial f}{\partial x_j}$$

$$\nabla \cdot \vec{F} = \sum_j \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x_j} \left(\frac{h_1 h_2 h_3}{h_j} F_j \right)$$

$$\nabla^2 f = \nabla \cdot \nabla f = \sum_j \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x_j} \left(\frac{h_1 h_2 h_3}{h_j^2} \frac{\partial f}{\partial x_j} \right)$$