Quiz #1, Problem 1

(a) $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

$$\Rightarrow \cos^2 x = \frac{1}{4}\left(e^{2ix} + 2 + e^{-2ix}\right) = \frac{1}{2} + \frac{1}{4}\underbrace{\left(e^{2ix} + e^{-2ix}\right)}_{2\cos 2x}$$

$$\boxed{\cos^2 x = \frac{1}{2}(1 + \cos 2x)}$$

$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

$$\Rightarrow \sin^2 x = -\frac{1}{4}\left(e^{2ix} - 2 + e^{-2ix}\right) = \frac{1}{2} - \frac{1}{4}\underbrace{\left(e^{2ix} + e^{-2ix}\right)}_{2\cos 2x}$$

$$\boxed{\sin^2 x = \frac{1}{2}(1 - \cos 2x)}$$

You could either of these from the other by using $\cos^2 x + \sin^2 x = 1$, and together the two imply that $\cos^2 x - \sin^2 x = \cos 2x$.

(b)
$$S_N = \sum_{n=0}^{N} R^{2n} \sin^2 n\theta$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} R^{2n}(1 - \cos 2n\theta)$$

$$= \frac{1}{2}\underbrace{\sum_{n=0}^{\infty} R^{2n}}_{= \frac{1 - R^{2(N+1)}}{1 - R^2}} - \frac{1}{2}\,\underbrace{\mathrm{Re}\left(\sum_{n=0}^{N} R^{2n} e^{2in\theta}\right)}_{= \mathrm{Re}\left(\frac{1 - (R^2 e^{2i\theta})^{N+1}}{1 - R^2 e^{2i\theta}}\right)}$$

$$= \mathrm{Re}\left(\frac{1 - (R^2 e^{2i\theta})^{N+1}}{1 - R^2 e^{2i\theta}} \cdot \frac{1 - R^2 e^{-2i\theta}}{1 - R^2 e^{-2i\theta}}\right)$$

$$= \mathrm{Re}\left(\frac{1 - R^{2(N+1)} e^{2i(N+1)\theta} - R^2 e^{-2i\theta} + R^{2(N+2)} e^{2iN\theta}}{1 - R^2 e^{2i\theta} - R^2 e^{-2i\theta} + R^4}\right)$$

$$= \mathrm{Re}\left(\frac{1 - R^{2(N+1)} e^{2i(N+1)\theta} - R^2 e^{-2i\theta} + R^{2(N+2)} e^{2iN\theta}}{1 - 2R^2 \cos 2\theta + R^4}\right)$$

$$\boxed{S_N = \frac{1}{2}\,\frac{1 - R^{2(N+1)}}{1 - R^2} - \frac{1}{2}\,\frac{1 - R^2\cos 2\theta - R^{2(N+1)}\cos 2(N+1)\theta + R^{2(N+2)}\cos 2N\theta}{1 - 2R^2\cos 2\theta + R^4}}$$

(c) When $N \to \infty$, the geometric series converges for $\boxed{R<1}$ and

$$\boxed{\lim_{N\to\infty} S_N = \frac{1}{2}\frac{1}{1-R^2} + \frac{1}{2}\frac{1-R^2\cos 2\theta}{1-2R^2\cos 2\theta + R^4}}$$