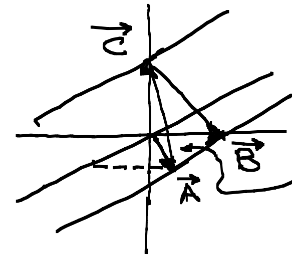
Quiz #1, Problem 2

$\vec{A} = \hat{x} + \hat{y}$

$\vec{B} = \hat{y}$

$\vec{C} = \hat{z}$



$\mathcal{M}$ is the plane that includes the line through the tips of $\vec{B}$ and $\vec{C}$ and extends parallel to the x axis in both directions.

(a) $\vec{A}-\vec{B}$ and $\vec{A}-\vec{C}$ lie in $\mathcal{M}$, so their cross product is orthogonal to $\mathcal{M}$. We normalize to get a unit vector

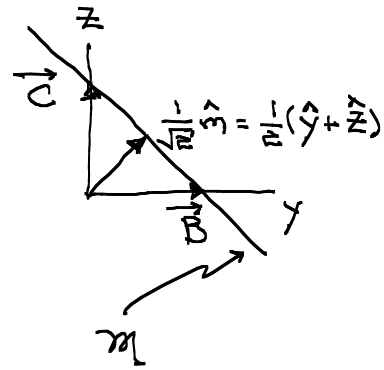
This is obvious from the picture.

$$\boxed{\hat{m} = \frac{(\vec{A}-\vec{B})\times(\vec{A}-\vec{C})}{|(\vec{A}-\vec{B})\times(\vec{A}-\vec{C})|} = \frac{1}{\sqrt{2}}(\hat{y}+\hat{z})}$$

The other possibility points in the opposite direction and has negative y and z components.

$$(\vec{A}-\vec{B})\times(\vec{A}-\vec{C}) = \underbrace{-\vec{A}\times\vec{C}}_{-\hat{y}+\hat{x}} + \underbrace{\vec{A}\times\vec{B}}_{\hat{z}} + \underbrace{\vec{B}\times\vec{C}}_{\hat{x}} = \hat{y}+\hat{z}$$

$$|(\vec{A}-\vec{B})\times(\vec{A}-\vec{C})| = (1+1)^{1/2} = \sqrt{2}$$



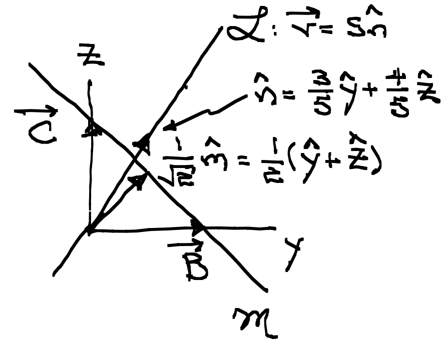
(b) $\mathcal{M}$ is defined by $\hat{m}\cdot\vec{r} = H$, so one can use $\vec{r} = \vec{A}, \vec{B}$, or $\vec{C}$ to find H.

$$\hat{m}\cdot\vec{A} = \frac{1}{\sqrt{2}}$$

$$\hat{m}\cdot\vec{B} = \frac{1}{\sqrt{2}} \Longrightarrow \boxed{H = \frac{1}{\sqrt{2}}}$$

$$\hat{m}\cdot\vec{C} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\hat{m} = \frac{1}{2}(\hat{y}+\hat{z})$$



(c) $\hat{n} = \frac{3}{5}\hat{y} + \frac{4}{5}\hat{z}$

We want the value of s such that

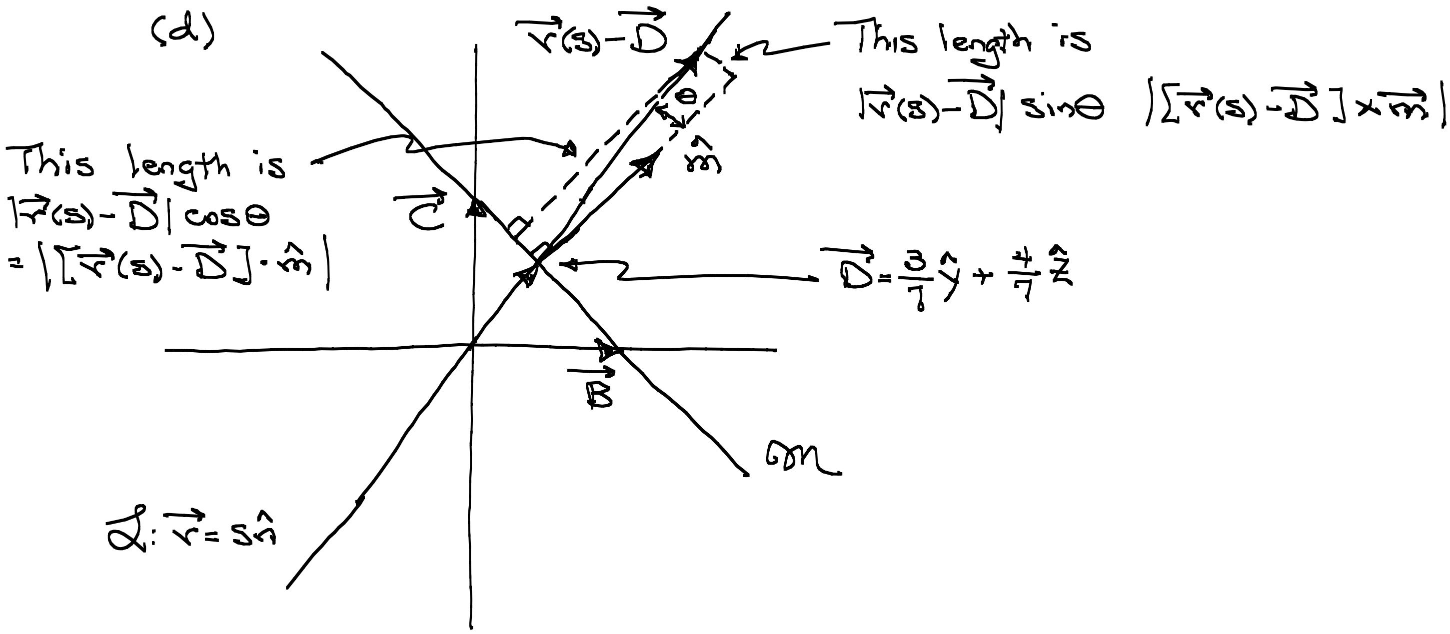
$$s\hat{n}\cdot\hat{m} = H \Longrightarrow \frac{1}{\sqrt{2}} = s\left(\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}}\right) = \frac{7s}{5\sqrt{2}}$$

$$\Longrightarrow s = \frac{5}{7}$$

So $\boxed{s = \frac{5}{7}, \quad \vec{D} = s\hat{n} = \frac{3}{7}\hat{y} + \frac{4}{7}\hat{z}}$

Vector on $\mathcal{L}$ with tip in $\mathcal{M}$
(intersection of $\mathcal{L}$ and $\mathcal{M}$)

(d)



$$\vec{r}(s) - \vec{D} = \frac{3s}{5}\hat{y} + \frac{4s}{5}\hat{z} - \frac{3}{7}\hat{y} - \frac{4}{7}\hat{z} = \left(\frac{s}{5} - \frac{1}{7}\right)(3\hat{y} + 4\hat{z})$$

$$d_1 = \left|[\vec{r}(s) - \vec{D}] \cdot \hat{m}\right| = \left|\left(\frac{s}{5} - \frac{1}{7}\right)\frac{1}{\sqrt{2}}(3+4)\right| = \boxed{\frac{1}{\sqrt{2}}\left|\frac{7s}{5} - 1\right| = d_1}$$

$$d_2 = \left|[\vec{r}(s) - \vec{D}] \times \hat{m}\right| = \left|\left(\frac{s}{5} - \frac{1}{7}\right)\frac{1}{\sqrt{2}}\underbrace{(3\hat{y} + 4\hat{z}) \times (\hat{y} + \hat{z})}_{3\hat{x} - 4\hat{x} = -\hat{x}}\right| = \boxed{\frac{1}{\sqrt{2}}\left|\frac{s}{5} - \frac{1}{7}\right| = d_2}$$

Either of these follows from the other by noting that

$$\left|\vec{r}(s) - \vec{D}\right|^2 = 25\left(\frac{s}{5} - \frac{1}{7}\right)^2$$