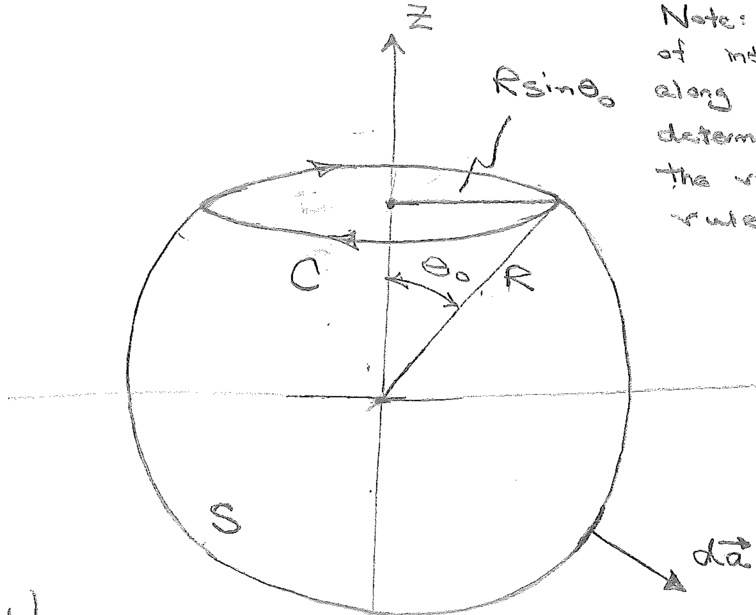
Quiz #2, Problem 2



Note: the direction of integration along C is determined by the right-hand rule.

$$\vec{A} = \hat{e}_r L r (3\cos^2\theta - 1) + \hat{e}_\theta M r \sin\theta\cos\theta + \hat{e}_\phi N r \sin\theta$$

(b)
$$\underbrace{\int \nabla\times\vec{A}\cdot d\vec{a} = \oint_C \vec{A}\cdot d\vec{l}}_{\text{Stokes's theorem}}$$

by right-hand rule, integrate in $-\phi$ direction

$$d\vec{l} = -R\sin\theta_0\, d\phi\, \hat{e}_\phi \quad (d\phi > 0)$$

$$\vec{A}\cdot d\vec{l} = -R\sin\theta_0\, d\phi\, \overbrace{A_\phi(R,\theta_0,\phi)}^{NR\sin\theta_0}$$

$$= -NR^2\sin^2\theta_0\, d\phi$$

$$\int \nabla\times\vec{A}\cdot d\vec{a} = -NR^2\sin^2\theta_0 \int_0^{2\pi} d\phi$$

$$= -2\pi N R^2 \sin^2\theta_0$$

$$\boxed{\int \nabla\times\vec{A}\cdot d\vec{a} = -2\pi N R^2 \sin^2\theta_0}$$

(a) Direct evaluation:

$$\int_S \nabla\times\vec{A}\cdot \vec{da} = \int_S R^2 \sin\theta \, d\theta \, d\phi \, (\nabla\times\vec{A})_r\Big|_{r=R,\theta,\phi}$$

$\vec{da} \to R^2 \sin\theta \, d\theta \, d\phi \, \hat{e}_r$

$$= R^2 \int_{\theta_0}^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, (\nabla\times\vec{A})_r\Big|_{r=R,\theta,\phi}$$

r- component of curl: $A_\phi = N r \sin\theta$ $\quad A_\theta = M r \sin\theta \cos\theta$

$$(\nabla\times\vec{A})_r = \frac{1}{r\sin\theta}\left[\underbrace{\frac{\partial}{\partial\theta}(\sin\theta A_\phi)} - \underbrace{\frac{\partial A_\theta}{\partial\phi}}_{=0}\right]$$

$$= \frac{\partial}{\partial\theta}(N r \sin^2\theta)$$

$$= N r \, 2\sin\theta\cos\theta$$

$$= \frac{2N\cancel{r\sin\theta}\cos\theta}{\cancel{r\sin\theta}}$$

$$= 2N\cos\theta$$

$$(\nabla\times\vec{A})_r\Big|_{r=R,\theta,\phi} = 2N\cos\theta \quad \text{or} \quad = \frac{N}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)$$

$$\int_S \nabla\times\vec{A}\cdot\vec{da} = R^2 \int_{\theta_0}^{\pi} \sin\theta \, d\theta \, 2N\cos\theta \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$N\frac{\partial}{\partial\theta}(\sin^2\theta)$

$$= 4\pi N R^2 \underbrace{\int_{\theta_0}^{\pi} \sin\theta\cos\theta \, d\theta}$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= \int_{\sin\theta_0}^{0} u \, du = \frac{1}{2}u^2\Big|_{\sin\theta_0}^{0} = -\frac{1}{2}\sin^2\theta_0$$

$$\boxed{\int_S \nabla\times\vec{A}\cdot\vec{da} = -2\pi N R^2 \sin^2\theta_0}$$

(c) Yet another approach: By Stokes's theorem, the integral over S is equal to the integral over the cap Y of the sphere, i.e., the surface $r=R$, $\theta\le\theta_0$, but with inward-directed area element $\vec{da} = -R^2\sin\theta\, d\theta\, d\phi\, \hat{e}_r$.

$\vec{da}$ points into the interior of the sphere

$$\int_S \nabla\times\vec{A}\cdot\vec{da} = \int_Y \nabla\times\vec{A}\cdot\underbrace{\vec{da}}_{-R^2\sin\theta\, d\theta\, d\phi\, \hat{e}_r}$$

$$= -R^2\int_Y \sin\theta\, d\theta\, d\phi\, (\nabla\times\vec{A})_r\Big|_{r=R,\theta,\phi}$$

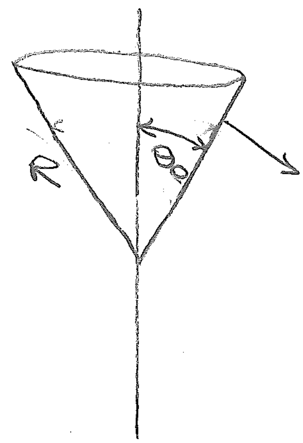
$$= -R^2\int_0^{\theta_0}\sin\theta\, d\theta\int_0^{2\pi} d\phi\, \underbrace{(\nabla\times\vec{A})_r\Big|_{r=R,\theta,\phi}}_{2N\cos\theta}$$

$$= -2NR^2\underbrace{\int_0^{\theta_0} d\theta\,\sin\theta\cos\theta}\ \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\int_0^{\sin\theta_0} u\,du = \frac{1}{2}u^2\Big|_0^{\sin\theta_0} = \frac{1}{2}\sin^2\theta_0$$

$$\boxed{\int_S \nabla\times\vec{A}\cdot\vec{da} = -2\pi N R^2\sin^2\theta_0}$$

Still another approach: By Stokes's theorem, the integral over S is equal to the integral over the cone $\theta = \theta_0$, with $r \leq R$.



$\vec{da} = \hat{e}_\theta \, dr \, r \sin\theta_0 \, d\phi$

$$\int_S \nabla \times \vec{A} \cdot \vec{da} = \int_{cone} \nabla \times \vec{A} \cdot \vec{da}$$

$$= \int_{cone} \underbrace{\nabla \times \vec{A} \cdot \hat{e}_\theta}\, r \sin\theta_0 \, dr \, d\phi$$

$$(\nabla \times \vec{A})_\theta = \frac{1}{r}\Big[\cancelto{0}{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi}} - \underbrace{\frac{\partial}{\partial r}(r A_\phi)}\Big]$$

$$= \frac{\partial}{\partial r}(N r^2 \sin\theta)$$

$$= 2 N r \sin\theta$$

$$= -2N \sin\theta$$

$$= -\int_0^R dr \int_0^{2\pi} d\phi \; 2 N r \sin^2\theta_0$$

$$= -2N \sin^2\theta_0 \underbrace{\int_0^R r\,dr}_{\frac{1}{2}R^2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= -2\pi N R^2 \sin^2\theta_0$$